Explicit Resolutions of Double Point Singularities of Surfaces

Alberto Calabri[*]   Rita Ferraro[†]

Abstract

Locally analytically, any isolated double point occurs as a double cover of a smooth surface. It can be desingularized explicitly via the canonical resolution, as it is very well-known. In this paper we explicitly compute the fundamental cycle of both the canonical and minimal resolution of a double point singularity and we classify those for which the fundamental cycle differs from the fiber cycle. Moreover we compute the conditions that a double point singularity imposes to pluricanonical systems.

1 Introduction

In this article, we give a detailed analysis of isolated surface singularities of multiplicity two, i.e., *double points*. Our goal is to be as explicit as possible.

A neighbourhood of an isolated double point p on a complex (normal) surface is analytically isomorphic to a double cover $\pi_0 : X_0 \to Y_0$ branched along a reduced curve B_0 with an isolated singularity at a point q_1, where Y_0 is a smooth surface. If x, y are local coordinates of Y_0 near q_1, then we may assume that X_0 is defined locally by an equation $z^2 = f(x, y)$, where B_0 is $f = 0$ and f is a square-free polynomial in x, y.

It is very well-known that one may desingularize $p = \pi_0^{-1}(q_1) \in X$ following the *canonical* resolution, which consists in desingularizing the branch curve and normalizing. This fact was implicitly used by the so-called Italian school (as one finds out for example by reading Castelnuovo and Enriques' papers on double covers of the projective plane), but it has been first written in 1946 by Franchetta [13], who computed the properties of the exceptional curves of the canonical resolution and the *fiber cycle*, that is the maximal effective divisor contained in the scheme-theoretic fiber over p.

Later, in 1978, Dixon [10] proved again Franchetta's results in a more modern language and he found sufficient conditions for the *fundamental cycle* of a double point singularity to be equal to the fiber cycle.

At the same time, Horikawa [16] and Laufer [19] explained the canonical resolution process too (see also [4]). In particular Laufer proved that the minimal resolution of a double point singularity is obtained from the canonical one by contracting simultaneously finitely many disjoint (-1)-curves and he described the relation between the topological types of the canonical resolution and of the minimal one, by using essentially the properties of the fundamental cycle.

We recall that the fundamental cycle of a resolution may well be computed inductively from the knowledge of the intersection matrix of the exceptional curves. However no explicit formula was known in the general case. In Theorem 11.2, we will give and prove such a formula, that turns out to be very simple, for the canonical resolution and then in (11.9) for the minimal one.

Moreover, this formula allows us also to classify those double points for which the fiber cycle strictly contains the fundamental cycle.

Finally in the last section we compute what are the conditions that a double point singularity imposes to canonical and pluricanonical systems of a surface. For this purpose, it is convenient to consider projective surfaces: we will assume that X_0 is a double plane, i.e., $Y_0 = \mathbb{P}^2$.

Although the general techniques used in this paper are very well-understood, we found no reference about our results. Thus we hope that this paper may serve as a natural complement to Dixon's and Laufer's papers and as an adequate reference for algebraic geometers that may use these results.

Moreover our approach is slightly different from the previous ones because we use, as discrete invariant of a plane curve singularity, the *Enriques* digraph, namely the directed graph involving the proximity relations among the infinitely near points to q_1, and we believe that this approach may be of independent interest as well (cf. Enriques diagrams in [15], [7] and [22]).

Thus we show also some examples of double point singularities which may help to understand the features of the combinatorial machinery we introduced. Please do not hesitate to contact the authors if you want to see the implementation of this approach on a computer.

The interested reader may also consult [3] for the analysis of an embedded resolution of $\{z^2 = f(x,y)\} \subset \mathbb{C}^3$.

2 Notation

To help the reader with the notation, which will soon become very heavy, we list here the main symbols used in this paper, together with the meaning and the reference formula or page where they are defined.

Symbol	Meaning	Ref.
X_0	normal complex surface	p. 4
p	isolated double point singularity of X_0	p. 4
Y_0	smooth surface	p. 4
$\pi_0 : X_0 \to Y_0$	double cover	p. 4
B_0	branch curve of π_0	p. 4
q_1	$= \pi_0(p)$, isolated singular point of B_0	p. 5
$\sigma_i : Y_i \to Y_{i-1}$	blowing-up at a point $q_i \in Y_{i-1}$	p. 6
q_i	center of the blowing-up σ_i	p. 6
$(\,\cdot\,)_i, \ (\,\cdot\,)$	intersection pairing in Y_i (resp. in Y_n)	p. 6
$\sigma_{ij} : Y_j \to Y_i$	$= \sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_{i+1}$	(5.1)
σ	$= \sigma_{0n}$, sequence of blowing-ups	p. 6
E_i	(proper transform of) the exceptional curve $\sigma_i^{-1}(q_i)$ ..	p. 6
E_i^*	total transform of E_i in $Y = Y_n$ via $\sigma = \sigma_{0n}$	p. 6
$N = (n_{ij})$	$n \times n$ matrix, $E_i = \sum_{j=1}^n n_{ij} E_j^*$	(5.3)
$M = (m_{ij})$	$n \times n$ matrix, $E_j^* = \sum_{k=1}^n m_{jk} E_k$	(5.3)
$q_j \to q_i$	q_j is proximate to q_i	p. 6
$q_j >^s q_i$	q_j is infinitely near of order s to q_i	p. 7
$Q = (q_{ij})$	$n \times n$ matrix, $q_{ij} = 1$ if and only if $q_j \to q_i$	p. 6
$S = (s_{ij})$	$n \times n$ matrix of intersection numbers $s_{ij} = (E_i \cdot E_j)$ ..	p. 7
$\tilde{B}_i$	proper transform of B_0 in Y_i	p. 7
$\tilde{\alpha}_i$	multiplicity of $\tilde{B}_{i-1}$ at q_i	(6.1)
$\tilde{\beta}_i$	$= \sum_{j=1}^n \tilde{\alpha}_i m_{ji}$	(6.2)
$\tilde{\Gamma}_i$	$= \tilde{B}_{n|E_i}$, divisor on E_i	p. 8
$\tilde{\gamma}_i$	$= \deg(\tilde{\Gamma}_i) = \tilde{B}_n \cdot E_i$	p. 8
$\pi_i : X_i \to Y_i$	normal double cover induced by π_0 and σ_{0i}	p. 10
B_i	branch curve of π_i	p. 10
μ_i	multiplicity of B_{i-1} at q_i	p. 10
ε_i	$= \mu_i \bmod 2 \in \{0, 1\}$, branchedness of E_i	(7.2)
α_i	$= \mu_i - \varepsilon_i$, even integer number	(8.1)
β_i	$= \sum_{j=1}^n \alpha_i m_{ji}$, even integer number	(8.1)
$\tau : X \to X_0$	canonical resolution of $p \in X_0$	p. 11
$\bar{\tau} : \bar{X} \to X_0$	minimal resolution of $p \in X_0$	p. 15
Y	$= Y_n$, smooth surface	p. 11
$\pi : X \to Y$	smooth double cover induced by the canonical resol...	p. 11
B	$= B_n$, smooth branch curve of π	p. 11
Γ_i	$= B_{|E_i}$, divisor on E_i	(8.4)
γ_i	$= \deg(\Gamma_i) = B \cdot E_i$	p. 11
F_i	$= \pi^*(E_i)$, exceptional curves of τ	p. 13
D	$= \sum_{i=1}^n (\alpha_i/2 - 1) E_i^*$	p. 14
D^*	$= \pi^* D = \tau^* K_{X_0} - K_X$, adjunction condition div.	(9.6)
F	fiber cycle of the canonical resolution	p. 14
$\bar{F}_i$	exceptional curves of $\bar{\tau}$	p. 16
$\bar{F}$	fiber cycle of the minimal resolution ....(10.6)
Z	fundamental cycle of the canonical resolution ....(11.1)
$\bar{Z}$	fundamental cycle of the minimal resolution ....(11.9)

Symbol	Meaning	Ref.
defective	q_i such that there exists $q_j >^1 q_i$ with $\alpha_j > \alpha_i$ .... p. 27
Def	index set of defective points .... p. 28

3 Double covers of surfaces

Throughout this paper a *double cover* of a smooth irreducible complex surface Y is a finite, surjective, proper holomorphic map $\pi : X \to Y$ of degree 2 branched along a reduced curve, where X is a (normal) irreducible complex surface. In the algebraic setting π is a finite morphism of degree 2. For more details on covers and double covers the reader may consult [4].

Let p be an isolated double point singularity of a surface. Since any isolated double point occurs, locally analytically, as a double cover of a smooth surface, let us consider as our initial data a double cover $\pi_0 : X_0 \to Y_0$ branched along the reduced curve B_0 defined in local coordinates by $f = 0$, where f is a square-free polynomial. If x, y are local analytic coordinates at a point $q \in Y_0$, then X_0 is defined by an equation $z^2 = f(x, y)$ and X_0 is normal.

If $q \notin B_0$, then $f(q) \neq 0$ and there are two pre-images of q in X_0; at each of these points X_0 is smooth and $\pi_0 : X_0 \to Y_0$ is unramified. If $q \in B_0$, then there is a single point of X_0 lying above q; X_0 is smooth at this point if and only if B_0 is smooth at q. The geometry of a smooth double cover is well-known:

Lemma 3.1 *Let Y be a smooth surface. Let $\pi : X \to Y$ be a double cover, branched along the smooth and reduced curve B in Y.*

1. *If C is an irreducible component of B, then $\pi_{|D} : D = \pi^{-1}(C) \to C$ is an isomorphism, $\pi^*(C) = 2D$ and $D^2 = C^2/2$.*
2. *If C is an irreducible curve in Y which meets B transversally in $2k$ points, where $k \geq 1$, then $D = \pi^{-1}(C)$ is a smooth irreducible curve on X and $\pi_{|D} : D \to C$ is a double cover branched along the $2k$ points of intersection of C with B. Moreover $\pi^*(C) = D$ and $D^2 = 2C^2$.*

Let E be a smooth rational curve in Y which is not part of the branch locus B. Let $\Gamma = B_{|E}$ be the divisor of intersection of B with E.

3. *If Γ is an* even *divisor, say $\Gamma = 2Q$ (in particular if $\Gamma = Q = 0$), then $\pi^{-1}(E) = D_1 + D_2$, where $\pi_{|D_i} : D_i \to E$ is an isomorphism and $D_i^2 = E^2 - \deg(Q)$ for $i = 1, 2$, and $D_1 \cdot D_2 = \deg(Q)$.*
4. *If Γ is not* even, *then $D = \pi^{-1}(E)$ is an irreducible curve, $\pi_{|D} : D \to E$ is a double cover and $D^2 = 2E^2$. D is singular at those points $q \in E$ where $\Gamma(q) \geq 2$; locally near $p = \pi^{-1}(q)$ the curve D has the analytic equation $z^2 = x^n$, where $n = \Gamma(q)$.*

Proof. One can check locally the properties of π. If C is not contained in B, then $\pi_{|\pi^{-1}(C)} : \pi^{-1}(C) \to C$ is a double cover branched along $B_{|C}$ and it is surely reducible if $B_{|C} = 2Q$ for some divisor $Q \neq 0$ on C. Since π has degree 2, we see that intersections double after applying π^*, i.e., for any divisors C_1 and C_2 in Y, $(\pi^*C_1 \cdot \pi^*C_2)_X = 2(C_1 \cdot C_2)_Y$, where $(\,\cdot\,)_X$ (resp. $(\,\cdot\,)_Y$) is the intersection form

in X (resp. Y). So all the claims about D^2 are trivial. Regarding again point 3, note that an unramified double cover of $\mathbb{P}^1$ is reducible, by Hurwitz formula (or the simply-connectedness of $\mathbb{P}^1$). □

In the assumption of Lemma 3.1, B is an even divisor and

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-B/2), \tag{3.2}$$

while Riemann-Hurwitz Formula is:

$$K_X = \pi^*(K_Y \otimes \mathcal{O}_Y(B/2)) \tag{3.3}$$

(see [4, Lemmas 17.1 and 17.2]). If $\pi : X \to Y$ is a double cover with X normal but not smooth, one can still define the canonical divisor K_X and the same formula (3.3) holds. For a normal surface X the canonical divisor K_X is defined as the Weil divisor class $\mathrm{div}(s)$, where s is a rational canonical differential. Since Weil divisors on a normal scheme do not depend on closed subsets of codim ≥ 2, one can easily verify that (3.3) holds by considering the smooth double cover $\pi_{|X_{sm}} : X_{sm} \to Y \setminus \mathrm{Sing}(B)$, where $X_{sm} = X \setminus \pi^{-1}(\mathrm{Sing}(B))$.

4 Resolving a singular double cover

Suppose that the branch curve B_0 of $\pi_0 : X_0 \to Y_0$ is not smooth at the point q_1, thus $p = \pi_0^{-1}(q_1)$ is a double point singularity of X_0.

Let us define $\mu_1 = \mathrm{mult}_{q_1}(B) = 2k + \varepsilon_1$, with $\varepsilon_1 \in \{0, 1\}$. In order to get a resolution of the singularity $p \in X_0$, we begin with resolving the branch locus B_0 of π_0. Let $\sigma : Y_1 \to Y_0$ be the blowing-up at $q_1 \in Y_0$, with exceptional curve E_1, and let $\tilde{B}_1$ be the proper transform of B_0 in Y_1.

Lemma 4.1 *Let X_1 be the double cover of Y_1 branched along $\tilde{B}_1 + \varepsilon_1 E_1$. Then X_1 is the normalization of the pullback $X_0 \times_{Y_0} Y_1$, and as such is both a double cover of Y_0 and dominates the singular surface X_0.*

Proof. The pullback $X_0 \times_{Y_0} Y_1$, is a double cover of Y_1 (using the second projection) branched along the pullback of B_0, which is $\pi^*(B_0) = \mu_1 E_1 + \tilde{B}_1$ (in fact it is defined by $z^2 = f(x, y)$ also). However in the local coordinates (u, v) of Y_1, where $x = u$ and $y = uv$, we have that $f(x, y) = f(u, uv) = u^{\mu_1} g(u, v)$ where $g(u, v)$ is a function whose series expansion is not divisible by u, and defines the proper transform $\tilde{B}_1$ of B_0. Then the pullback is defined by $z^2 = u^{\mu_1} g(u, v)$, and is not normal if $\mu_1 \geq 2$; if $\mu_1 = 2k + \varepsilon_1$ with $\varepsilon \in \{0, 1\}$, then $w = z/u^k$ satisfies the monic equation $w^2 = u^{\varepsilon_1} g(u, v)$. The normalization X_1 of the pullback is defined by this monic equation, and is clearly a double cover of Y_1 branched along $\tilde{B}_1 + \varepsilon_1 E_1$. □

This normal surface X_1 dominates X_0, via the first projection, and gives a partial resolution of the double point singularity. We have passed to the double cover $X_1 \to Y_1$, and may iterate the procedure, continuing to blow up the branch curve at each of its singular points, then normalizing the double cover equation. It is already known that this process eventually terminates in a smooth double

cover $X_n \to Y_n$ that is called the *canonical* resolution of X_0 (see Theorem 7.4). Lemma 4.1 says that the series of double covers are determined by the parities of the multiplicities of the singular points involved. If the multiplicity is even, then the exceptional curve is not part of the new branch locus, only the proper transform; if the multiplicity is odd, then the exceptional curve is part of the new branch locus. This is all well-known, see for example [4]. We note that the multiplicity μ_1 may be determined on Y_1 by $\mu_1 = \tilde{B}_1 \cdot E$.

5 Blowing up a smooth surface

Let us consider a sequence of blowing-ups of a smooth surface Y_0, each one at a single point. We fix a particular order for the blowing-ups, and let Y_i be the surface obtained after the i-th blowing-up $\sigma_i : Y_i \to Y_{i-1}$ at a point $q_i \in Y_{i-1}$. Let σ_{ij} be the composition of the blowing-up maps from Y_j to Y_i:

$$\sigma_{ij} = \sigma_j \circ \cdots \circ \sigma_{i+2} \circ \sigma_{i+1} : Y_j \to Y_i, \tag{5.1}$$

for $0 \le i < j \le n$, where n is the total number of blowing-ups. Set $\sigma = \sigma_{0n} : Y_n \to Y_0$, $Y = Y_n$ and $(\cdot)_i$, resp. $(\cdot)$, the intersection form in Y_i, resp. in Y_n. The exceptional curve $E_i = \sigma_i^{-1}(q_i)$ in Y_i satisfies $(E_i \cdot E_i)_i = -1$ and $(E_i \cdot \sigma_i^*(C))_i = 0$ for any divisor C of Y_{i-1}. We will abuse notation and refer to the proper transform of E_i on $Y_{j \ge i}$ also as E_i, so $E_1, \dots, E_n$ are the *exceptional curves* for σ. Let $E_i^* = \sigma_{in}^*(E_i)$ be the total transform of E_i in Y_n via σ_{in}.

It is well-known that the relative Picard group $\mathrm{Pic}(Y_n)/\sigma^* \mathrm{Pic}(Y)$ is freely generated by the classes $\{E_i\}_{1 \le i \le n}$, as well as by $\{E_j^*\}_{1 \le j \le n}$, and the latter ones are an *orthonormal* basis in the sense that:

$$(E_i^* \cdot E_j^*) = -\delta_{ij}, \tag{5.2}$$

where δ is the Kronecker delta. Therefore we may write:

$$E_i = \sum_{j=1}^n n_{ij} E_j^*, \qquad E_j^* = \sum_{k=1}^n m_{jk} E_k \tag{5.3}$$

where the matrix $M = (m_{jk})$ is the inverse of $N = (n_{ij})$. Let $Q = (q_{ij})$ be the strictly upper triangular matrix defined by $q_{ij} = 1$ if q_j lies on E_i and $q_{ij} = 0$ otherwise. Following the classical terminology, we say that the point q_j is *proximate* to q_i, and we write $q_j \to q_i$, if and only if $q_{ij} = 1$.

Lemma 5.4 *Let I be the identity matrix. Then $M = I + Q + \cdots + Q^{n-1}$ and*

$$N = I - Q. \tag{5.5}$$

Proof. The first formula follows from (5.5), because $Q^m = 0$ for $m \ge n$. We prove (5.5) by induction on n. For $n = 1$, it is clear. If (5.5) holds for $n-1$, then the proper transform of E_i in Y_{n-1} is $\sigma_{i,n-1}^*(E_i) - \sum_{j=i+1}^{n-1} q_{ij}\sigma_{j,n-1}^*(E_j)$, so the proper transform of E_i in Y_n is $\sigma_n^*(E_i) - q_{in}E_n = \sigma_{in}^*(E_i) - \sum_{j=i+1}^{n-1} q_{ij}\sigma_{jn}^*(E_j) - q_{in}E_n^*$ and we conclude comparing with the first formula in (5.3). $\square$

More explicitly, M can be computed inductively from Q as follows. Suppose that the first $n-1$ columns of M are known. If q_{in} is the unique non-zero entry of the last column of Q, then the last column of M is equal to the i-th column of M (apart $m_{nn} = 1$), otherwise there is also $q_{jn} = 1$ and the last column of M is the sum of the i-th and the j-th column of M (except $m_{nn} = 1$). Let us consider the matrix Q as the adjacency matrix of a directed graph G, that we call the *Enriques* digraph of σ: the vertices of G are the points q_i, for $i = 1, \dots, n$, and there is an arrow from q_j to q_i if and only if $q_{ij} = 1$ (i.e. q_j is proximate to q_i).

Remark 5.6 *The properties of Q imply that an Enriques digraph is characterized by the following four properties (see [5, §5], [15, pp. 213–214]):*

   i) *there is no directed cycle;*
   ii) *every vertex has* out-degree *at most 2;*
   iii) *if $q_i \to q_j$ and $q_i \to q_k$, with $j \neq k$, then either $q_j \to q_k$ or $q_k \to q_j$;*
   iv) *there is at most one q_i with $q_i \to q_j$ and $q_i \to q_k$, if $j \neq k$.*

Note that the *in-degree* of q_i (the number of arrows ending in q_i) is $-E_i^2 - 1$.

Since we need only to resolve an isolated singularity at q_1, we assume to blow up only points lying on the total exceptional divisor, i.e., we assume that $q_i \in \sigma_{1,i-1}^{-1}(q_1)$, for every $i > 1$. This means that only the first column Q_1 of Q is everywhere zero, so the Enriques digraph is connected. Recall that a point q_j is called *infinitely near* to q_i, and we write $q_j > q_i$, if $q_j \in \sigma_{i,j-1}^{-1}(q_i)$. Thus each q_i is infinitely near to q_1. Let us define the *infinitesimal order* inductively. If $q_j > q_i$ and there is no q_k such that $q_j > q_k > q_i$, then q_j is infinitely near of *order one* to q_i and we write $q_j >^1 q_i$. If $q_j >^1 q_k > q_i$, then by induction $q_k >^m q_i$ for some m and we set $q_j >^{m+1} q_i$.

Usually, the main combinatorial tool used for blowing-ups is the dual graph of the exceptional curves and their self-intersection numbers, that are the entries of the intersection matrix $S = (s_{ij})$, where $s_{ij} = (E_i \cdot E_j)$.

Lemma 5.7 *The configuration of the exceptional curves E_i of σ may be given by only one of the following matrices: M, N, Q, or S. Indeed anyone of them determines canonically all the others.*

Proof. Recall that $N = M^{-1} = I - Q$. Formulas (5.3) and (5.2) imply that:

$$s_{ij} = \left(\sum_{k=1}^n n_{ik} E_k^* \cdot \sum_{h=1}^n n_{jh} E_h^* \right) = \sum_{k=1}^n \sum_{h=1}^n n_{ik} n_{jk} (E_k^* \cdot E_h^*) = -\sum_{k=1}^n n_{ik} n_{jk},$$

so $S = -NN^T = N(-I)N^T$, that is the decomposition of S in an unipotent upper triangular, a diagonal and an unipotent lower triangular matrix. Such a decomposition is known to be unique by linear algebra. $\qquad \square$

6   The proper transform of the singular curve

We now consider a reduced curve B_0 on Y_0 with a singular point at the point $q_1 \in Y_0$ which is being blown up. Let us denote with $\tilde{B}_i$ the proper transform of

B_0 in Y_i via the sequence of blowing-ups σ_{0i}, for every $i = 1, \ldots, n$. Recall that the following formula holds in $\operatorname{Pic} Y_n$:

$$\sigma^*(B_0) = \tilde{B}_n + \sum_{i=1}^{n} \tilde{\alpha}_i E_i^*, \qquad \text{where } \tilde{\alpha}_i = \operatorname{mult}_{q_i}(\tilde{B}_{i-1}). \tag{6.1}$$

Abusing language a little, usually $\tilde{\alpha}_i$ is called the *multiplicity* of B_0 at q_i. Note that $\tilde{\alpha}_i$ may be determined also in Y_i by $\tilde{\alpha}_i = (\tilde{B}_i \cdot E_i)_i$.

On the other hand, in $\operatorname{Pic} Y_n$ we may write also:

$$\sigma^*(B_0) = \tilde{B}_n + \sum_{i=1}^{n} \tilde{\beta}_i E_i. \tag{6.2}$$

for some non-negative integers $\tilde{\beta}_i$. Putting the second formula of (5.3) in (6.1) we find that the $\tilde{\beta}_i$'s can be computed from the $\tilde{\alpha}_i$'s as follows:

$$\tilde{\beta}_i = \sum_{j=1}^{n} \tilde{\alpha}_j m_{ji}, \qquad \text{or shortly} \qquad \tilde{\beta} = \tilde{\alpha} M, \tag{6.3}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are row vectors with the obvious entries.

The quantities $\tilde{\alpha}$ and $\tilde{\beta}$ may also be determined on Y_n knowing how $\tilde{B}_n$ intersects the exceptional curves E_i. Indeed, intersecting (6.2) with E_i gives $0 = \tilde{B}_n \cdot E_i + \sum_{j=1}^{n} \tilde{\beta}_j E_i \cdot E_j$ that is, setting $\tilde{\gamma}_j = (\tilde{B}_n \cdot E_j)$ and $\tilde{\gamma}$ the corresponding row vector:

$$\tilde{\gamma} = -\tilde{\beta} S = \tilde{\beta} N N^T = \tilde{\alpha} N^T, \qquad \text{so} \qquad \tilde{\alpha} = \tilde{\gamma} M^T. \tag{6.4}$$

Note that $\tilde{B}_n$ satisfies $(\tilde{B}_n \cdot E_i) \geq 0$ for every i, which is equivalent by (6.1), (5.3), (5.5) and (5.2) to the so-called *proximity inequality* at q_i:

$$\tilde{\alpha}_i \geq \sum_{j=1}^{n} q_{ij} \tilde{\alpha}_j = \sum_{j : q_j \to q_i} \tilde{\alpha}_j. \tag{6.5}$$

Suppose that B_0 has only an isolated singularity in q_1 and that $\sigma : Y_n \to Y_0$ resolves the singularities of B_0, i.e., the proper transform $\tilde{B}_n$ is smooth. Then the topological type of the singularity of B_0 at q_1 is completely determined by the matrix M, which carries the configuration of the exceptional curves E_i for σ, and the intersection divisor $\tilde{\Gamma}_i = \tilde{B}_{n|E_i}$, which says how $\tilde{B}_n$ meets E_i, for every $i = 1, \ldots, n$. Each $\tilde{\Gamma}_i$ is a non-negative divisor on E_i and if q is a point of intersection of two exceptional curves E_i and E_j, then:

$$\tilde{\Gamma}_i(q) = 0 \iff \tilde{\Gamma}_j(q) = 0. \tag{6.6}$$

Moreover if these numbers are non-zero, then at least one number is equal to one. Condition (6.6) says that either $\tilde{B}_n$ passes through q or not, while the latter statement that $\tilde{B}_n$ cannot be tangent in q to both exceptional curves simultaneously. These divisors $\tilde{\Gamma}_i$ express the combinatorial information of the singularity of the

branch curve completely. Given the configuration of the exceptional curves, they can be arbitrary, subject to the above condition.

We remark that the knowledge of M and of the degrees $\tilde{\gamma}_i = \deg(\tilde{\Gamma}_i)$ is equivalent to the knowledge of Q and the multiplicities $\tilde{\alpha}_i$ of B_0 at q_i, by (6.4).

Therefore we will define the *weighted Enriques digraph* of $q_1 \in B_0$ by attaching to each vertex q_i of the Enriques digraph the weight $\tilde{\alpha}_i = \mathrm{mult}_{q_i}(\tilde{B}_{i-1})$.

Example 6.7 *Let $B_0 \subset Y_0$ be defined locally near the origin $q_1 = (0,0)$ by:*

$$x(y^2 - x)(y^2 + x)(y^2 - x^3)(y^2 + x^3) = 0.$$

Clearly B_0 has multiplicity $\tilde{\alpha}_1 = 7$ at q_1. Blow up q_1: $\tilde{B}_1 \subset Y_1$ has two singular points q_2 and q_3 on E_1 of multiplicity $\tilde{\alpha}_2 = 3$ and $\tilde{\alpha}_3 = 2$ (q_2 and q_3 are infinitely near points of order one to q_1). Then blow up q_2 and q_3. $\tilde{B}_2 \subset Y_2$ meets transversally E_2 and it is smooth at those points, but $\tilde{B}_3 \subset Y_3$ has in $q_4 = E_1 \cap E_3$ a point of multiplicity $\tilde{\alpha}_4 = 2$, so q_4 is proximate to both q_3 and q_1, but q_4 is infinitely near of order 2 to q_1. Classically, q_4 is called a *satellite* point to q_1. Finally $\tilde{B}_4 \subset Y_4$ is smooth.
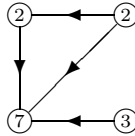
The configuration of the exceptional curves is determined by anyone of the following matrices:

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

and the combinatorial data of $\tilde{B}_4$ by anyone of the following vectors:

$$\tilde{\alpha} = (7, 3, 2, 2), \qquad \tilde{\beta} = (7, 10, 9, 18), \qquad \tilde{\gamma} = (0, 3, 0, 2),$$

that we encode in the following weighted Enriques digraph:

7   Resolving the branch locus of a double cover

We return to consider a normal double cover $\pi_0 : X_0 \to Y_0$ branched along the singular reduced curve B_0. We desingularize B_0 by successive blowing-ups as in the previous sections. We have seen that σ induces a normal double cover $\pi_n : X_n \to Y_n$, which is the normalization of the pullback $X_0 \times_{Y_0} Y_n$. As in Lemma 4.1, π_n is branched along a reduced curve B_n, obtained from $\sigma^*(B_0)$ by removing the multiple components an even number of times. Hence B_n is made of the proper transform $\tilde{B}_n$ of B_0 and possibly of some exceptional curves E_i. We set ε_i equal to one or zero, depending on whether E_i is part of the branch

locus of Y_n or not (and we say that E_i is *branched*, resp. *unbranched*). Setting ε the corresponding row vector, by (6.2) and (6.3) we have that:

$$\varepsilon = \tilde{\beta} \bmod 2 \quad \text{and} \quad \varepsilon = \tilde{\alpha} M \bmod 2. \tag{7.1}$$

Note that the branchedness of E_i is determined at the moment E_i is created on Y_i by blowing up the point $q_i \in Y_{i-1}$. Indeed, Lemma 4.1 says that $\varepsilon_i = 1$ (resp. $\varepsilon_i = 0$) if the multiplicity μ_i at q_i of the branch locus B_{i-1} of $\pi_{i-1} : X_{i-1} \to Y_{i-1}$ is odd (resp. even). Shortly:

$$\varepsilon = \mu \bmod 2. \tag{7.2}$$

Assuming inductively to know ε_j for $j < i$, the multiplicity of B_{i-1} at q_i is

$$\mu_i = \tilde{\alpha}_i + \sum_{j=1}^{i-1} \varepsilon_i q_{ji} = \tilde{\alpha}_i + \sum_{j:q_i \to q_j} \varepsilon_j, \qquad \text{or shortly} \qquad \mu = \tilde{\alpha} + \varepsilon Q. \tag{7.3}$$

If we have blown up to make the proper transform $\tilde{B}_n$ smooth, we still may not have the total branch locus B_n smooth. Singularities of the total branch locus now come from two sources: intersections of $\tilde{B}_n$ with branched E_i's, and intersections between two different branched E_i's. We first take up the former case. Suppose that $\tilde{B}_n$ meets the exceptional configuration $\bigcup_i E_i$ at a point q_{n+1}. We blow up q_{n+1} to create a new surface Y_{n+1}, a new proper transform $\tilde{B}_{n+1}$, and a new exceptional curve E_{n+1}. Since $\tilde{B}_n$ is smooth at q_{n+1}, we have $(\tilde{B}_{n+1} \cdot E_{n+1})_{n+1} = 1$. We now have new intersection divisors $\tilde{\Gamma}'_i$ for each $i = 1, \dots, n+1$. These are related to the previous intersection divisors $\tilde{\Gamma}_i$'s for $i = 1, \dots, n+1$ as follows:

$$\tilde{\Gamma}'_i = \begin{cases} \tilde{\Gamma}_i & \text{if } q_{n+1} \notin E_i \\ \tilde{\Gamma}_i - q_{n+1} & \text{if } q_{n+1} \in E_i \end{cases}$$

for every $i = 1, \dots, n$ and $\tilde{\Gamma}'_{n+1} = q$, where q is the point of intersection of $\tilde{B}_{n+1}$ with E_{n+1}. We may now iterate this construction, and arrive at the situation (increasing the number of blowing-ups n) that the proper transform $\tilde{B}_n$ does not meet any branched exceptional curves, i.e., the new exceptional divisors $\tilde{\Gamma}'_i$ are zero for each i such that $\varepsilon_i = 1$. Finally if any two exceptional curves now meet, we simply blow up the point of intersection once, and obtain an unbranched exceptional curve which now separates the two branched exceptional curves.

At this point we have a nonsingular total branch locus, hence a smooth double cover. We note that we still have the matrices M, N, Q, S for the current configuration of exceptional curves and the numbers $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$ and ε_i, defined for each i, as before. All the above process gives a proof of the following theorem (cf. [4, III.§6], [19, Theorem 3.1]):

Theorem 7.4 (The canonical resolution) *Let $\pi_0 : X_0 \to Y_0$ be a double cover with X_0 normal and Y_0 smooth. Then there exists a birational morphism*

$\sigma : Y \to Y_0$ such that the normalization X of the pullback $X_0 \times_{Y_0} Y$ is smooth. Moreover $\pi : X \to Y$ is a double cover and the diagram

$$
\begin{array}{ccc}
X & \overset{\tau}{\longrightarrow} & X_0 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\
Y & \overset{\sigma}{\longrightarrow} & Y_0
\end{array}
\qquad (7.5)
$$

commutes. So $\tau : X \to X_0$ is a resolution of singularities of X_0. □

We say that $\tau : X \to X_0$ (toghether with the double cover map $\pi : X \to Y$) is the *canonical* resolution of the double cover $\pi_0 : X_0 \to Y_0$, because X and Y are unique, up to isomorphism, assuming that the centers of the blowing-ups $\sigma_i : Y_i \to Y_{i-1}$, which factorizes σ, are always singular points of the branch curve of $X_{i-1} \to Y_{i-1}$. We will see in section 10 that the canonical resolution might not be minimal. However, we may think the canonical resolution as the "minimal" resolution in the category of double covers over smooth surfaces.

8   The branch curve of the canonical resolution

As in the previous section, let $\tau : X \to X_0$ be the canonical resolution of X_0 and $\pi : X \to Y$ be the smooth double cover. Let us write B for the branch curve of $\pi : X \to Y$ and suppose that $\sigma : Y \to Y_0$ factorizes in n blowing-ups $\sigma_i : Y_i \to Y_{i-1}$, with $Y_n = Y$, so the exceptional curves for σ are $E_1, \dots, E_n$ and all the formulas in the previous sections hold for $B_n = B$ and $\tilde{B}_n = \tilde{B}$. The branch curve B can be written in $\operatorname{Pic} Y$ as:

$$
B = \tilde{B} + \sum_{i=1}^{n} \varepsilon_i E_i = \sigma^*(B_0) - \sum_{i=1}^{n} \beta_i E_i = \sigma^*(B_0) - \sum_{i=1}^{n} \alpha_i E_i^* \qquad (8.1)
$$

for some non negative integers β_i and α_i. Comparing (8.1) with (6.2), (6.1) and (5.3) we see that:

$$
\beta = \tilde{\beta} - \varepsilon \qquad \text{and} \qquad \alpha = \tilde{\alpha} + \varepsilon N = \tilde{\alpha} + \varepsilon(Q - I). \qquad (8.2)
$$

Moreover, from formulas (8.2), (6.3) and (7.3) we find that:

$$
\alpha = \mu - \varepsilon \qquad \text{and} \qquad \alpha = \beta N. \qquad (8.3)
$$

By (8.2) and (7.1), (8.3) and (7.2) it follows that the β_i's and the α_i's are all even. From (7.3) and (8.2) it is clear that $\alpha = \mu = \tilde{\alpha}$ if and only if $\varepsilon = 0$, that happens if $\tilde{\alpha}_i$ is even for every $i = 1, \dots, n$, i.e. if B_0 has even multiplicity at each singular point, including the infinitely near ones.

In order to measure how the branch curve B of the canonical resolution meets the exceptional curves, let us introduce the following intersection divisors:

$$
\Gamma_i = B_{|E_i} = \tilde{B}_{|E_i} + \sum_{j:\varepsilon_j = 1} E_{j|E_i} \qquad (8.4)
$$

and set $\gamma_i = \deg(\Gamma_i)$. By definition $\Gamma_i = 0$ if E_i is branched. However, Γ_i could be zero even if $\varepsilon_i = 0$. We claim that the Γ_i's have the following properties:

1. Γ_i is a non-negative divisor and γ_i is even;
2. if $q = E_i \cap E_j$ and $\varepsilon_i = \varepsilon_j = 0$, then $\Gamma_i(q) = 0$ if and only if $\Gamma_j(q) = 0$. If these numbers are non-zero, then at least one of them is equal to 1.
3. if $q = E_i \cap E_j$, $\varepsilon_i = 1$ and $\varepsilon_j = 0$, then $\Gamma_j(q) = 1$.

It suffices to show that γ_i is even, since all the other properties of Γ_i are induced by those of $\tilde{\Gamma}_i$, which we have already seen in section 6. If E_i is branched, then $\gamma_i = 0$ and the thesis is trivial. Otherwise, if E_i is unbranched, then:

$$\gamma_i = \tilde{\gamma}_i + \sum_{j \neq i} \varepsilon_j (E_i \cdot E_j) \equiv \tilde{\gamma}_i + \sum_{j=1}^{n} \tilde{\beta}_j s_{ij} \bmod 2$$

hence $\gamma \equiv \tilde{\gamma} + \tilde{\beta} S \bmod 2$ and the claim follows from $\tilde{\gamma} = -\tilde{\beta} S$ (see (6.4)).

In order to encode the combinatorial data of the double point singularity $p = \pi_0^{-1}(q_1) \in X_0$, we define the *weighted Enriques digraph* of p by attaching to each vertex q_i of the Enriques digraph of $q_1 \in B_0$ the weight $\mu_i = \mathrm{mult}_{q_i}(B_{i-1})$.

In the next example we will illustrate in detail how the canonical resolution process goes on.

Example 8.5 *Let B_0 be a curve defined locally near the origin $q_1 = (0,0)$ by:*

$$y(y^2 - x^3) = 0.$$

Clearly, we need just two blowing-ups to smooth the proper transform $\tilde{B}_2$ of B_0. However, we have to blow up 5 more times in order to get a smooth double cover. In fact E_1, E_2 are branched and $\tilde{B}_2$ passes through $q_3 = E_1 \cap E_2$ and meets E_2 also in another point q_4. Thus $B_2 = \tilde{B}_2 + E_1 + E_2$ has multiplicity $\mu_3 = 3$ at q_3, hence $\varepsilon_3 = 1$ and $\tilde{B}_3$ meets E_3 in a point q_5. Now B_3 has only nodes in $q_4, q_5, q_6 = E_1 \cap E_3$ and $q_7 = E_2 \cap E_3$, therefore $E_4, \ldots, E_7$ are unbranched and B_7 is smooth. Our combinatorial data are:

$$M = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{aligned} \tilde{\alpha} &= (3,2,1,0,0,1,1) \\ \tilde{\gamma} &= (0,0,0,1,1,0,0), \\ \mu &= (3,3,3,2,2,2,2), \\ \varepsilon &= (1,1,1,0,0,0,0), \\ \alpha &= (2,2,2,2,2,2,2), \\ \gamma &= (0,0,0,2,2,2,2), \end{aligned}$$

that we encode in the Enriques digraph, weighted with the μ_i's (see the right-hand side graph of Figure 1). For the readers' convenience, we inserted in Figure 1 (on the left-hand side) also the Enriques digraph weighted with the $\tilde{\alpha}_i$'s and labelled with the q_i's.

Figure 1 may help to understand formula (7.3), namely how to compute inductively the μ_i's (thus the ε_i's) from the $\tilde{\alpha}_i$'s. Start from q_1: the weight of q_1 is 3, thus $\mu_1 = 3$, $\varepsilon_1 = 1$ and add 1 to all the weights attached to vertices with arrows ending in q_1 (namely q_2, q_3 and q_6). Now consider q_2: the actual weight of q_2 is 3, hence $\mu_2 = 3$, $\varepsilon_2 = 1$ and add 1 to the weights of q_3, q_4 and q_7, which are the vertices with arrows ending in q_2. Then go on inductively, for all the q_i's. Clearly, no change of weights is made at step i if $\varepsilon_i = 0$.
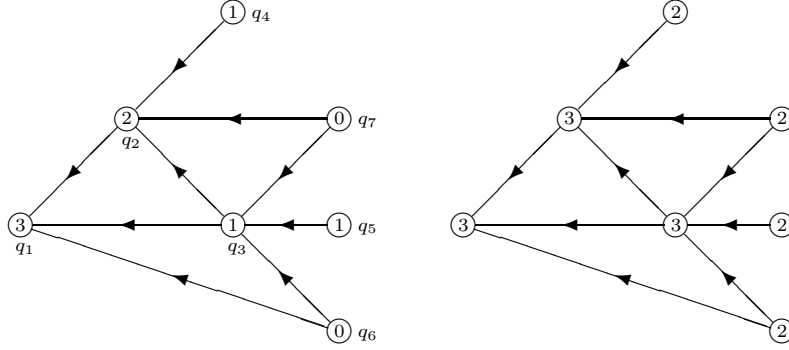
Figure 1: The Enriques digraph weighted resp. with the $\tilde{\alpha}_i$'s and the μ_i's

9 The description of the canonical resolution

The description of the canonical resolution of the singularity on X_0 is now a combinatorial problem, using the information of the configuration of the exceptional curves E_i (described by the matrix M or the Enriques digraph) and the divisors $\tilde{\Gamma}_i$ (subject to the conditions stated in section 6).

We then can compute the quantities $\tilde{\gamma}$, $\tilde{\alpha}$, μ, ε, and determine (from ε) which of the E_i's are branched curves, and finally determine the divisors Γ_i.

We now apply Lemma 3.1 for each exceptional curve. If $\pi : X \to Y$ is the double cover map, let us define $F_i = \pi^{-1}(E_i)$ for each i, thus $F_1, \ldots, F_n$ are all the exceptional curves for the canonical resolution $\tau : X \to X_0$.

Remark 9.1 *A curve F_i is reducible if and only if $\varepsilon_i = 0$ and $\Gamma_i = B_{|E_i}$ is an even divisor. In that case, F_i splits in two smooth rational curves F_i' and F_i'', with $F_i' \cdot F_i'' = \gamma_i/2$ and $F_i'^2 = F_i''^2 = E_i^2 - \gamma_i/2$.*

If $\varepsilon_i = 0$, then $\pi^*(E_i) = F_i$, otherwise, if $\varepsilon_i = 1$, then $\pi^*(E_i) = 2F_i$, i.e.

$$\pi^*(E_i) = (1 + \varepsilon_i)F_i. \tag{9.2}$$

Moreover since intersections double after applying π^* we have that:

$$F_i^2 = \frac{2}{(1 + \varepsilon_i)^2}E_i^2 \qquad \text{and} \qquad (F_i \cdot F_j) = (2 - \varepsilon_i - \varepsilon_j)(E_i \cdot E_j). \tag{9.3}$$

We claim that the arithmetic genus of F_i is, for each i:

$$p_a(F_i) = \frac{\gamma_i}{2} + \varepsilon_i - 1. \tag{9.4}$$

If $\varepsilon_i = 1$, then F_i is a smooth rational curve, $\Gamma_i = 0$ and the claim is trivial. If F_i splits, (9.4) follows from Remark 9.1. Otherwise, F_i is a double cover of E_i branched along Γ_i and (9.4) is just Hurwitz formula.

Moreover F_i is singular at a point P if and only if $\Gamma_i(\pi(P)) > 1$, thus in particular F_i is smooth at the intersection points with F_j, for each $j \neq i$.

Now we want to find an explicit formula for the canonical divisor K_X. By Riemann-Hurwitz formula (3.3) we know that $K_X = \pi^*(K_Y + B/2)$, where $K_Y = \sigma^*(K_{Y_0}) + \sum_i E_i^*$. Therefore by (8.1):

$$K_Y + \frac{B}{2} = \sigma^* \left(K_{Y_0} + \frac{B_0}{2} \right) - \sum_{i=1}^{n} \left(\frac{\alpha_i}{2} - 1 \right) E_i^*. \qquad (9.5)$$

We define $D = \sum_{i=1}^{n}(\alpha_i/2 - 1)E_i^*$, so:

$$K_X = (\sigma \circ \pi)^*(K_{Y_0} + B_0/2) - \pi^*D = \tau^* K_{X_0} - D^*, \qquad (9.6)$$

where $D^* = \pi^*D$ is called the *adjunction condition* divisor (cf. section 14). We remark that $D^* \geq 0$, because $\alpha_i \geq 2$ for each i (since in the canonical resolution process we blow up only singular points of the branch curve).

Let us show now the explicit formula for the *fiber cycle* F of the canonical resolution, already written without proof by Franchetta and Dixon. The *fiber cycle* of τ is the maximal effective divisor F contained in the *scheme theoretic fiber* of τ, i.e., the subscheme of X defined by the inverse image ideal sheaf $\tau^{-1}m_{p,X_0}$ of the maximal ideal m_{p,X_0} of p in X_0. Therefore:

$$F = \gcd \{ \mathrm{div}(g) \,|\, \text{for all } g \in \tau^{-1}m_{p,X_0} \}.$$

For example, the fiber cycle of the sequence $\sigma : Y \to Y_0$ of blowing-ups is E_1^*.

Theorem 9.7 (Franchetta) *The fiber cycle of the canonical resolution is:*

$$F = \pi^*(E_1^*) = \sum_{i=1}^{n} m_{1i}(1 + \varepsilon_i)F_i. \qquad (9.8)$$

Proof. The second equality in (9.8) follows from (9.2) and from the definition of M (see (5.3)), so it suffices to show the first equality. Recall that locally x, y are the coordinates of Y_0 near $q_1 = (0,0)$, X_0 is defined by $z^2 = f(x,y)$ and π_0 is the projection $(x,y,z) \mapsto (x,y)$. Hence x, y and z are generators of the ideal sheaf $\tau^{-1}m_{p,X_0}$ as $\mathcal{O}_{X_0,p}$-module. Therefore F is the greatest effective divisor contained in the pullback divisors $\tau^* \mathrm{div}\, x$, $\tau^* \mathrm{div}\, y$ and $\tau^* \mathrm{div}\, z$. From the commutativity of the diagram (7.5) we see that $\tau^* \mathrm{div}\, x = (\pi_0 \circ \tau)^* \mathrm{div}\, x = (\sigma \circ \pi)^* \mathrm{div}\, x$ and $\tau^* \mathrm{div}\, y = (\sigma \circ \pi)^* \mathrm{div}\, y$, thus the gcd of the divisors $\tau^* \mathrm{div}\, x$ and $\tau^* \mathrm{div}\, y$ is the pullback $\pi^*(E_1^*)$ of the fiber cycle E_1^* of σ. Moreover $\tau^* \mathrm{div}\, z^2 = (\pi_0 \circ \tau)^* \mathrm{div}\, f = (\sigma \circ \pi)^* \mathrm{div}\, f$, thus the divisor $2\tau^* \mathrm{div}\, z = \tau^* \mathrm{div}\, z^2$ is equal to the pullback $\pi^*(\sigma^* B_0)$ of the total transform $\sigma^* B_0$ which contains E_1^* with multiplicity $\tilde{\alpha}_1 \geq 2$. So $\tau^* \mathrm{div}\, z \supseteq \pi^*(E_1^*)$, hence $F = \pi^*(E_1^*)$. $\qquad \square$

The self-intersection of the fiber cycle is:

$$F^2 = \pi^*(E_1^*) \cdot \pi^*(E_1^*) = 2E_1^{*2} = -2 = -\mathrm{mult}_p(X_0). \qquad (9.9)$$

By (9.8) and (9.2), the fiber cycle F has the following properties:

$$F \cdot F_1 = (2 - \varepsilon_1)E_1^* \cdot E_1 = -2 + \varepsilon_1, \quad F \cdot F_i = (2 - \varepsilon_i)E_1^* \cdot E_i = 0 \qquad (9.10)$$

for every $i > 1$. Thus the normal sheaf $\mathcal{O}_{F_i}(F_i)$ of F_i, for $i > 1$ (that can be useful if $p_a(F_i) > 0$), is given by how F_i meets the other components of F:

$$m_{1i}(1 + \varepsilon_i)F_{i|F_i} = -F^{\hat{i}}_{|F_i} \tag{9.11}$$

where $F^{\hat{i}} = F - m_{1i}(1 + \varepsilon_i)F_i$. Finally, by (9.6), (9.8) and (9.9), the arithmetic genus of the fiber cycle is:

$$p_a(F) = (F \cdot K_X + F^2)/2 + 1 = E_1^* \cdot (K_Y + B/2) = \alpha_1/2 - 1. \tag{9.12}$$

10 The minimal resolution

It may happen that the canonical resolution $\tau : X \to X_0$ of $p \in X_0$ is not minimal. However the following theorem (cf. [19, Th. 5.4]) shows that the canonical resolution is not too far to be minimal.

Theorem 10.1 *Let $\tau : X \to X_0$ be the canonical resolution and $\bar{\tau} : \bar{X} \to X_0$ the minimal one. Then $\tau = \tau' \circ \bar{\tau}$, where $\tau' : X \to \bar{X}$ is the blowing-up at finitely many distinct points. Moreover none of these points is singular for the exceptional curves of $\bar{\tau}$ and neither lies on the intersection (necessarily transverse) of more than three of them.*

The following lemma characterizes the (-1)-curves in $\tau^{-1}(p)$ and allows us to give an elementary proof of Theorem 10.1, easier than Laufer's original one.

Lemma 10.2 *An exceptional curve F_j for τ is a (-1)-curve if and only if $F_j = \pi^{-1}(E_j)$, where E_j is branched and $E_j^2 = -2$. Moreover the (-1)-curves in $\tau^{-1}(p)$ are disjoint.*

Proof. Let E_j be unbranched. By Lemma 3.1, if $\Gamma_j = B_{|E_j}$ is not even, then F_j cannot be a (-1)-curve, because $F_j^2 = 2E_j^2$ would be even. If Γ_j is even, then $\pi^{-1}(E_j) = F_j' + F_j''$ and $F_j' = F_j'' = E_j^2 - \deg(\Gamma_j/2)$, thus F_j' (and F_j'') could be a (-1)-curve only if $E_j^2 = -1$ and $\Gamma_j = 0$, that means that q_j was unnecessarily blown up. Hence we are left only with the possibility that E_j is branched and $E_j^2 = 2F_j^2 = -2$. Since no branched exceptional curves meet in Y, neither do two (-1)-curves in X. $\qquad\square$

Remark 10.3 *If $F_j = \pi^{-1}(E_j)$ is a (-1)-curve in X, then μ_j is odd and there is exactly one q_i such that $q_i >^1 q_j$ and $\mu_i = \mu_j + 1$. Usually one says that B_{j-1} has two infinitely near points of the same, odd, multiplicity μ_j at q_j.*

Proof. Clearly μ_j is odd (and > 2), because $\varepsilon_j = 1$. Since E_j is a (-2)-curve, we blew up only one point q_i on E_j: this means that all the intersections of $\tilde{B}_j$ with E_j are supported on q_i, i.e. $\tilde{\alpha}_i = \tilde{\alpha}_j$, and the thesis follows from (7.3). $\quad\square$

Let us denote by $\bar{\tau} : \bar{X} \to X_0$ the minimal resolution of $p \in X_0$.

Proof of Theorem 10.1. Let $\tau' : X \to \tilde{X}$ be the contraction of all the (-1)-curves in $\tau^{-1}(p)$. We claim that $\tilde{X}$ is isomorphic to $\bar{X}$ and $\tau = \tau' \circ \bar{\tau}$, namely

there is no (-1)-curve in $\tau'(\tau^{-1}(p))$. The only way for a (-1)-curve to be created after blowing down a (-1)-curve $F_j = \pi^{-1}(E_j)$ is for a smooth rational curve F_k of self-intersection -2 to meet the given (-1)-curve F_j. Since no two branched curve meet on Y and E_j is branched by Lemma 10.2, then F_k must lie over an unbranched curve E_k which meets E_j at one point. Therefore $\pi^*(E_k)$ cannot split (its divisor Γ_k is not even) so that $F_k = \pi^*(E_k)$ and $F_k^2 = 2E_k^2$. So E_k is a (-1)-curve on Y and, since F_k is smooth and rational, the divisor Γ_k must consist of two simple points (one of which is the intersection point with E_j). In this case we see that E_k and E_j should be blown down on Y, so that both of these curves were unnecessarily blown up. This proves our claim. Let $F_j = \pi^{-1}(E_j)$ be a (-1)-curve of X. Let E_k be an exceptional curve that meets E_j. Since E_j is branched, E_k is unbranched and their intersection is transversal, as any intersection of the E_i's. Hence F_j contracts to a smooth point of F_k (cf. Lemma 3.1). Finally, $E_j^2 = -2$ implies that E_j meets at most three exceptional curves of σ, namely one curve that corresponds to a blown up point on E_j and the exceptional curves on which q_j lies, that are at most two. $\qquad\square$

The previous analysis offers also an alternative route to obtaining the minimal resolution of $p \in X_0$: we may first contract the branched (-2)-curves among the E_i's and then take the double cover, namely the diagram

$$
\begin{array}{ccc}
X & \overset{\tau'}{\longrightarrow} & \bar{X} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}} \\
Y & \overset{\sigma'}{\longrightarrow} & \bar{Y}
\end{array}
$$

commutes, where σ' is the contraction of the branched (-2)-curves in $\sigma^{-1}(q_1) \subset Y$ and X is the fiber product $\bar{X} \times_{\bar{Y}} Y$.

Note that contracting a (-2)-curve on a smooth surface produces a singularity, namely an ordinary double point (of type A_1, see p. 24). Since the (-2)-curves are branched for $\pi : X \to Y$, the singular points of $\bar{Y}$ must be considered as *branched points* for $\bar{\pi} : \bar{X} \to \bar{Y}$.

Let us denote by $\bar{F}_i$ (resp. $\bar{E}_i$) the image of F_i in $\bar{X}$ (resp. of E_i in $\bar{Y}$). For simplicity, suppose that we blow down only a (-1)-curve F_k (the general case can be computed inductively). If E_k meets two unbranched divisors E_i and E_j in X, then $\bar{E}_i$ and $\bar{E}_j$ meets in a branched point, hence:

$$
\bar{F}_i \cdot \bar{F}_j = \begin{cases} 1 & \text{if } E_i \cdot E_k = E_j \cdot E_k = 1, \\ F_i \cdot F_j & \text{if } E_i \cdot E_k = 0 \text{ or } E_j \cdot E_k = 0. \end{cases} \tag{10.4}
$$

Moreover the self-intersection numbers change as follows:

$$
\bar{F}_i^2 = \begin{cases} F_i^2 + 1 & \text{if } E_i \cdot E_k = 1, \\ F_i^2 & \text{otherwise,} \end{cases} \tag{10.5}
$$

while the arithmetic genera stay unchanged. Formula (9.8) and the fact that $\tau'^{-1}(\bar{\tau}^{-1}(m_{p,X_0})) = \tau^{-1}(m_{p,X_0})$ imply that the fiber cycle of the minimal resolution $\bar{\tau} : \bar{X} \to X$ is:

$$
\bar{F} = \sum m_{1i}(1 + \varepsilon_i)\bar{F}_i. \tag{10.6}
$$

11   The fundamental cycle

The fundamental cycle of the (canonical) resolution is the unique smallest positive cycle:

$$Z = \sum_{i=1}^{n} z_i F_i \tag{11.1}$$

with $z_i > 0$ such that $Z \cdot F_k \leq 0$ for every $k = 1, \dots, n$. If F_i splits in F_i' and F_i'' as in Remark 9.1, *a priori* we should consider in (11.1) two distinct coefficients z_i' and z_i''. But if z_i' were different from z_i'' in (11.1), then we could exchange them and take the g.c.d. of the cycles, so $z_i = \min\{z_i', z_i''\}$ would fulfill the properties of the fundamental cycle. Therefore we may and will assume $z_i' = z_i'' = z_i$.

In general the fundamental cycle of any resolution can be computed inductively as follows. Let $F_1, \dots, F_n$ be the exceptional curves.

(1) Set $Z = \sum_{i=1}^{n} F_i$.
(2) Check if $Z \cdot F_j \leq 0$ for every j.
(3) If (2) is false, there exists j such that $Z \cdot F_j > 0$. Replace Z with $Z + F_j$ and go back to (2).
(4) Otherwise, if (2) is true, Z is the fundamental cycle.

In [17] Laufer used essentially the properties of the fundamental cycle in order to describe precisely the relation between the topological types of the canonical and the minimal resolution. However Laufer showed only an implicit formula for Z (see Lemma 5.5 in [17]). In the next theorem we give an explicit formula for Z, that turns out to be very simple and that may help to understand better the algorithms described by Laufer in [17, Theorems 5.7 and 5.10]. It is natural to compare the fiber cycle with the fundamental one. The definitions implies that $F \geq Z$. It is known that the equality holds for every resolution for special types of singularities, for example rational [2] and minimally elliptic [18, Theorem 3.13] ones. Regarding double points, Dixon showed that $Z = F$ for every resolution if $\mu_1 = \mathrm{mult}_{q_1}(B_0)$ is even [10, Theorem 1], and that $Z = F$ for the minimal resolution if B_0 is analytically irreducible at q_1 [10, Theorem 2]. In section 12 we will classify all the double point singularities for which $F > Z$.

Theorem 11.2 *The fundamental cycle Z of the canonical resolution $\tau : X \to X_0$ of $p \in X_0$ differs from the fiber cycle $F = \pi^*(E_1^*)$ if and only if there exists $j > 1$ such that $\varepsilon_j = 0$, q_j is infinitely near of order one to q_1 and:*

$$m_{1i} + m_{ji} \text{ is even for every } i \text{ such that } \varepsilon_i = 0. \tag{11.3}$$

In that case the fundamental cycle is:

$$Z = \frac{1}{2}\pi^*(E_1^* + E_j^*) = \sum_{i=1}^{n} \frac{1}{2}(1 + \varepsilon_i)(m_{1i} + m_{ji})F_i. \tag{11.4}$$

We remark that condition (11.3) implies that $\varepsilon_1 = 1$, because $m_{11} = 1$ and $m_{j1} = 0$, thus the multiplicity $\tilde{\alpha}_1 = \mu_1$ of B_0 at q_1 is odd. Furthermore j is

uniquely determined. Indeed, if $q_i >^1 q_1$ (and $i \neq j$), then $m_{1i} = 1$ and $m_{ji} = 0$, so (11.3) would imply that $\varepsilon_i = 1$.

Proof. By (9.10), $Z \leq F$. If $Z \neq F$, we may write:

$$F = Z + P \tag{11.5}$$

where $P = \sum_i t_i F_i$ is a positive (non-zero) divisor. It follows from (9.9) that:

$$-2 = F^2 = Z^2 + P^2 + 2Z \cdot P.$$

Since $Z > 0$ and $P > 0$, then $Z^2 < 0$ and $P^2 < 0$. Moreover $Z \cdot P \leq 0$ because Z is the fundamental cycle, so the only possibility is:

$$Z^2 = P^2 = -1, \qquad Z \cdot P = 0.$$

Hence $F \cdot P = -1$ and by formulas (9.10), one finds that: $-1 = F \cdot P = F \cdot t_1 F_1 = -(2 - \varepsilon_1)t_1$, which forces $t_1 = \varepsilon_1 = 1$, so E_1 must be branched and F_1 is forced to belong to P with multiplicity one. Since $Z \cdot F_k \leq 0$ for each k, $Z^2 = -1$ implies that there exists an unique j such that:

$$Z \cdot F_j = -1, \qquad Z \cdot F_k = 0 \quad \text{for every } k \neq j,$$

and $z_j = 1$. Therefore $t_j = -Z \cdot P = 0$, i.e., F_j is not a component of P, and the coefficient of F_j in F is $(1 + \varepsilon_j)m_{1j} = z_j + t_j = 1$, so

$$\varepsilon_j = 0 \quad \text{and} \quad m_{1j} = 1,$$

in particular $j \neq 1$. It follows from (11.5) that: $P \cdot F_1 = -1$, $P \cdot F_j = 1$, $P \cdot F_k = 0$ for $k \neq 1, j$. The previous three equations are equivalent to:

$$\sum_{i=1}^{n}(2 - \varepsilon_i)t_i s_{ik} = \begin{cases} -2 & \text{if } k = 1 \\ 1 & \text{if } k = j \\ 0 & \text{if } k \neq 1, j. \end{cases}$$

Recalling that:

$$\sum_{i=1}^{n} 2m_{1i}s_{ik} = 2(E_1^* \cdot E_k) = \begin{cases} -2 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$$

and setting m_1 and t row vectors with the obvious entries, one finds that:

$$((2 - \varepsilon)t - 2m_1)S = e_k$$

where e_k is the row vector with the k-th entry equal to 1 and 0 everywhere else. Multiplying both sides with S^{-1}, the vector $((2 - \varepsilon)t - 2m_1)$ is the k-th row of the matrix S^{-1}. In particular: $2t_j - 2m_{1j} = 0 - 2 = -2$ is the (j, j)-entry in $S^{-1} = -M^t M$. Therefore:

$$-2 = -\sum_{1 \leq i \leq j} m_{ij}^2 = -m_{1j}^2 - m_{jj}^2 - \sum_{1 < i < j} m_{ij}^2 = -2 - \sum_{1 < i < j} m_{ij}^2$$

that is possible if and only if $m_{ij} = 0$ for every $1 < i < j$. This means that q_j is proximate to q_1, but $q_j \not> q_i$ for $i \neq 1$, i.e. $q_j >^1 q_1$. Furthermore, the (k, j)-entry in S^{-1} is:

$$(2 - \varepsilon_k)t_k - 2m_{1k} = - \sum_{1 \leq i \leq j} m_{ik} m_{ij} = -m_{1k} - m_{jk},$$

that we may rewrite as follows: $t_k = \frac{1}{2}(1 + \varepsilon_k)(m_{1k} - m_{jk})$. Since the coefficient of F_k in F is $(1 + \varepsilon_k)m_{1k} = t_k + z_k$, then

$$z_k = \frac{1}{2}(1 + \varepsilon_k)(m_{1k} + m_{jk}) \tag{11.6}$$

must be an integer, that proves (11.3) and (11.4).

Vice versa, if there exists j as in the statement, we may order the blowing-ups σ_i in such a way that $j = 2$. It suffices to show that $Z' = \pi^*(E_1^* + E_2^*)/2$ has the property that $Z' \cdot F_k \leq 0$ for every $k = 1, \dots, n$. Indeed $Z' < F$, so the first part of the proof implies that Z' has to be the fundamental cycle. If $k > 2$, then: $Z' \cdot F_k = (1 - \varepsilon_k/2)(E_1^* \cdot E_k + E_2^* \cdot E_k) = 0$. Moreover $Z' \cdot F_2 = E_2^{*2} = -1$ and $Z' \cdot F_1 = E_1^* \cdot E_1/2 + E_2^* \cdot E_1/2 = 0$. □

If $F > Z$, the arithmetic genus of Z is, by $Z^2 = -1$, (11.4) and (9.5):

$$\begin{aligned} p_a(Z) &= \frac{E_1^* + E_j^*}{2} \cdot \left(\sigma^* \left(K_{Y_0} + \frac{B_0}{2} \right) - \sum_{k=1}^{n} \left(\frac{\alpha_k}{2} - 1 \right) E_k^* \right) + \frac{1}{2} = \\ &= \frac{\alpha_1 + \alpha_j - 2}{4} = \frac{\tilde{\alpha}_1 + \tilde{\alpha}_j - 2}{4}. \end{aligned} \tag{11.7}$$

where the last equality follows from $\alpha_1 = \tilde{\alpha}_1 - 1$ and $\alpha_j = \tilde{\alpha}_j + 1$.

Now we compute the fundamental cycle of the minimal resolution.

Lemma 11.8 *Let $\bar{\tau} : \bar{X} \to X_0$ be the minimal resolution of $p \in X_0$. Then the fundamental cycle $\bar{Z}$ of $\bar{\tau}$ is:*

$$\bar{Z} = \sum_i z_i \bar{F}_i \tag{11.9}$$

where $\bar{F}_i$ are the exceptional curves of $\bar{\tau}$ and $Z = \sum_i z_i F_i$ is the fundamental cycle of the canonical resolution.

Proof. Without any loss of generality, we may assume to blow down only a (-1)-curve $F_k = \pi^{-1}(E_k)$, where $E_k^2 = -2$ and $\varepsilon_k = 1$. Recall that E_k meets at least one and at most three unbranched divisors. First, we shall prove that:

$$\sum_{i \neq k} z_i \bar{F}_i \cdot \bar{F}_j \leq 0. \tag{11.10}$$

for every $j \neq k$. For this purpose, we claim that:

$$z_k = \sum_{i : E_i \cdot E_k = 1} z_i. \tag{11.11}$$

Suppose that E_k meets three unbranched divisors, i.e., $q_k = E_{k_1} \cap E_{k_2}$ and we blew up a point q_{k_3} lying on E_k, with $\varepsilon_{k_i} = 0$ for $i = 1, 2, 3$. Then $m_{1k} = m_{1k_3} = m_{1k_1} + m_{1k_2}$, so:

$$2m_{1k} = (1 + \varepsilon_k)m_{1k} = \sum_{i=1}^{3}(1 + \varepsilon_{k_i})m_{1k_i} = m_{1k_1} + m_{1k_2} + m_{1k_3}$$

which proves (11.11) if $Z = F$, by (9.8). Similarly, $2m_{2k} = m_{2k_1} + m_{2k_2} + m_{2k_3}$ and (11.11) holds even if $Z < F$, by (11.6). If E_k meets only one or two unbranched divisors, there are four possible configurations and the proof of (11.11) is analogous.

Clearly (11.10) holds if $E_j \cdot E_k = 0$. Otherwise if $E_j \cdot E_k = 1$, by (10.4), (10.5) and the fact that if $E_j \cdot E_k = E_i \cdot E_k = 1$ then $F_i \cdot F_j = 0$, formula (11.11) implies that:

$$\sum_{i \neq k} z_i \bar{F}_i \cdot \bar{F}_j = \sum_{i:E_i \cdot E_k = 1} z_i \bar{F}_i \cdot \bar{F}_j + \sum_{i:E_i \cdot E_k = 0} z_i F_i \cdot F_j =$$
$$= \sum_{i:E_i \cdot E_k = 1} z_i + \sum_{i \neq k} z_i F_i \cdot F_j = z_k + \sum_{i \neq k} z_i F_i \cdot F_j = Z \cdot F_j \leq 0$$

which proves (11.10). Let $\bar{Z} = \sum_{i \neq k} s_i \bar{F}_i$ be the fundamental cycle of $\bar{\tau}$. If we show that for every j:

$$\left(\sum_{i \neq k} s_i F_i + \sum_{i:E_i \cdot E_k = 1} s_i F_k \right) \cdot F_j \leq 0, \tag{11.12}$$

then $s_i = z_i$, for $i \neq k$, and (11.9) holds. Indeed if $E_j \cdot E_k = 0$ then (11.12) is trivial. If $E_j \cdot E_k = 1$, then the left hand side of (11.12) becomes:

$$\sum_{i:E_i \cdot E_k = 0} s_i F_i \cdot F_j + \sum_{i:E_i \cdot E_k = 1} s_i F_i \cdot F_j + \sum_{i:E_i \cdot E_k = 1} s_i F_k \cdot F_j = \sum_{i \neq k} s_i \bar{F}_i \cdot \bar{F}_j = \bar{F} \cdot \bar{F}_j \leq 0.$$

Finally, for $j = k$, the left hand side of (11.12) is:

$$\sum_{i \neq k} s_i F_i \cdot F_k + \sum_{i:E_i \cdot E_k = 1} s_i F_k^2 = \sum_{i:E_i \cdot E_k = 1} s_i F_i \cdot F_k - \sum_{i:E_i \cdot E_k = 1} s_i = 0. \qquad \square$$

Corollary 11.13 *Let F, Z (resp. $\bar{F}, \bar{Z}$) be the fiber and the fundamental cycle of the canonical (resp. minimal) resolution. Then $F > Z$ and $\bar{F} = \bar{Z}$ if and only if q_2 is the unique proximate point to q_1 and $\tilde{\alpha}_1 = \tilde{\alpha}_2$ is odd. Furthermore, this happens if and only if $\bar{F}^2 = \bar{Z}^2 = -1$.*

Proof. Suppose that $F > Z$ and $\bar{F} = \bar{Z}$. This means that F_1 is a (-1)-curve that we blow down, hence $E_1^2 = -2$ and there is only one proximate point to q_1, that is q_2, so $\tilde{\alpha}_1 = \tilde{\alpha}_2$. Moreover $\tilde{\alpha}_1$ is odd by Theorem 11.2. Conversely, if E_1 is branched and $E_1^2 = -2$, then F_1 is a (-1)-curve that we blow down. Hence

$m_{2i} = m_{1i}$ for every $i > 2$, therefore the coefficient of F_i in F is the same as the coefficient of F_i in Z, for every $i > 2$. The last assertion follows from the fact that if the fundamental cycle of a resolution of p has self-intersection -2, then on any resolution the fundamental cycle is equal to the fiber cycle (cf. [19, Lemma 5.2] or [10, p. 110]). $\qquad\square$

12 The description of the Enriques digraph

We want to describe the weighted Enriques digraph of those double point singularity for which the fundamental cycle of the canonical resolution is strictly contained in the fiber cycle. Recall that the weight of the vertex q_i is μ_i, i.e. the multiplicity at q_i of the branch curve B_{i-1} of $\pi_{i-1} : X_{i-1} \to Y_{i-1}$. Before going on, we need some remark about proximate points.

Let us call *proximity subgraph* of q_1 the subgraph of the Enriques digraph consisting only of the proximate points to q_1 (and the arrows among them).

We may order the σ_i's (the blowing-ups) in such a way that $q_2, \ldots, q_{n'}$ are all the proximate points to q_1 and for every $j = 1, \ldots, h$:

$$q_{i_j} >^1 q_{i_j-1} >^1 \cdots >^1 q_{i_{j-1}+2} >^1 q_{i_{j-1}+1} >^1 q_1, \qquad (12.1)$$

where $n' = i_h > i_{h-1} > \cdots > i_2 > i_1 > i_0 = 1$. Thus the proximity digraph of q_1 has the shape of Figure 2, which looks like a *flower* with h petals. We say that the j-th petal has *length* $i_j - i_{j-1}$.

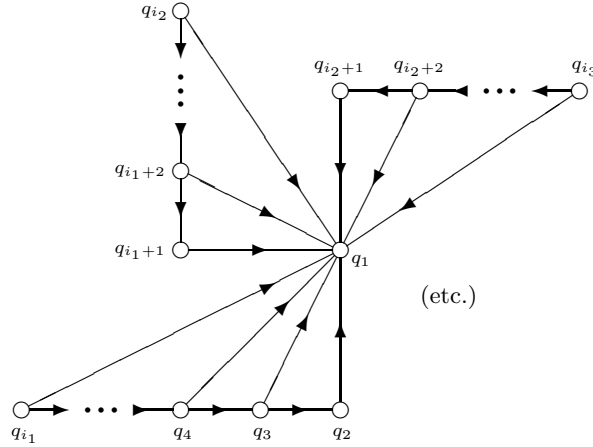Clearly the proximity subgraph of any point has a similar shape.

Figure 2: The proximity subgraph of q_1

Let us say that a vertex of the Enriques digraph is *very odd* if its weight is odd, its proximity subgraph has exactly one petal of odd length and all the other petals of even length. Now we are ready to prove the following:

Theorem 12.2 *The fundamental cycle is strictly contained in the fiber cycle of the canonical resolution of $p \in X_0$ if and only if the weighted Enriques digraph of $q_1 = \pi(p)$ has the following properties:*

1. *q_1 is a very odd vertex (in particular its weight μ_1 is odd);*
2. *a proximate point q_i to q_1, belonging to a petal of even (resp. odd) length of the proximity subgraph of q_1, is a very odd vertex if and only if q_i is infinitely near of odd (resp. even) order to q_1;*
3. *inductively, 1 and 2 hold replacing q_1 with any very odd vertex.*

Proof. Suppose that the fundamental cycle Z is strictly contained in the fiber cycle F. By Theorem 11.2, μ_1 is odd and there exists j such that $\varepsilon_j = 0$, $q_j >^1 q_1$ and condition (11.3) holds. Moreover we may and will assume that $j = 2$, so $\varepsilon_2 = 0$ and μ_2 is even.

Consider the proximity subgraph of q_1 as above (cf. formula (12.1) and Figure 2). We claim that i_j is even for every $j = 1, \ldots, h$, thus q_1 is a very odd vertex.

Suppose that $i_1 > 2$, namely in the canonical resolution process we blow up $q_3 = E_2 \cap E_1$. Then $m_{13} = 2$ and $m_{23} = 1$, so condition (11.3) implies that $\varepsilon_3 = 1$ and μ_3 is odd. Since $\varepsilon_1 = \varepsilon_3 = 1$, the intersection $q_4 = E_3 \cap E_1$ is a singular point of the branch curve B_3 of $\pi_3 : X_3 \to Y_3$, so we must blow up also q_4. Similarly, if we blow up $q_5 = E_4 \cap E_1$, then $m_{15} + m_{25} = 5$, thus condition (11.3) forces $\varepsilon_5 = 1$ and we must blow up also $q_6 = E_5 \cap E_1$. Repeating this argument, it follows that the first petal has odd length and i_1 is even.

Look at the second petal. Now $q_{i_1+1} >^1 q_1$, so $m_{1,i_1+1} = 1$ and $m_{2,i_1+1} = 0$. Hence (11.3) implies that $\varepsilon_{i_1+1} = 1$ and μ_{i_1+1} is odd. Therefore we must blow up $q_{i_1+2} = E_{i_1+1} \cap E_1$. If $i_2 > i_1 + 2$, it means that we blow up also $q_{i_1+3} = E_{i_1+2} \cap E_1$, then $m_{1,i_1+3} + m_{2,i_1+3} = 3$, so (11.3) forces that $\varepsilon_{i_1+3} = 1$ and we must blow up $q_{i_1+4} = E_{i_1+3} \cap E_1$ too. Proceeding in this way, this shows that the second petal has even length and i_2 is even. The same argument works for the j-th petal, with $j > 2$, just by replacing i_1 with i_{j-1}. This proves our claim that q_1 is a very odd vertex.

Now we want to show that $\varepsilon_i \equiv i \pmod 2$ for every $i = 1, \ldots, i_h$, and q_{2l-1} is a very odd vertex for every $l = 2, \ldots, i_h/2$.

We already know that $\varepsilon_2 = 0$ and $\varepsilon_{2i-1} = 1$ for every $i = 1, \ldots, i_h/2$. Suppose by contradiction that $\varepsilon_4 = 1$ (and $i_h > 2$). Since $\varepsilon_3 = 1$, we must blow up also $q_k = E_4 \cap E_3$ and $m_{1k} + m_{2k}$ is odd, so condition (11.3) implies that $\varepsilon_k = 1$. Hence we must blow up $q_{k+1} = E_k \cap E_4$ too, and $m_{1,k+1} + m_{2,k+1}$ is again odd, thus $\varepsilon_{k+1} = 1$ by (11.3). Going on in this way, we produce each time another branched exceptional curve, so we should never stop blowing up, contradicting Theorem 7.4. This shows that $\varepsilon_4 = 0$ and μ_4 is even. The proof that $\varepsilon_{2l} = 0$ for every $l = 3, \ldots, i_h/2$ is similar.

Consider the proximity subgraph of q_{2l-1}, for $l = 2, \ldots, i_h/2$. Repeating exactly the same arguments as for q_1 and q_2, one finds out that q_{2l} (which is proximate to q_{2l-1}) belongs to a petal of odd length, while all other petals of the proximity subgraph of q_{2l-1} have even length, thus q_{2l-1} is a very odd vertex.

It remains to prove that the proximity subgraph of a non-very-odd vertex can be arbitrary. For every $l = 1, \ldots, i_h/2$, we proved that $\varepsilon_{2l} = 0$, so q_{2l} cannot be very odd. Moreover $m_{1,2l} + m_{2,2l}$ is even. If q_k is proximate to q_{2l} (and

$k \neq 2l + 1$), then $m_{1k} + m_{2k}$ is a multiple of $m_{1,2l} + m_{2,2l}$, hence it is even and (11.3) imposes no condition on q_k. This means that the proximity subgraph of a non-very-odd vertex, as q_{2l}, can be arbitrary and concludes the proof that the Enriques digraph has properties 1, 2 and 3.

Conversely, suppose that the three properties hold. One may easily check that the m_{ij}'s satisfy condition (11.3), where the wanted q_j is the infinitely near point of order one to q_1 belonging to the petal of odd length, therefore one concludes by Theorem 11.2. $\qquad\square$

Note that, with the notation of the proof, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are odd, while $\tilde{\alpha}_i$ is even for every $i = 3, \dots, i_h$, by (7.3). Moreover $\varepsilon_1 = 1$ forces $\tilde{B} \cdot E_1 = 0$, or equivalently $\tilde{\alpha}_1 = \sum_{j=1}^{n} \tilde{\alpha}_j q_{1j} = \sum_{j=2}^{i_h} \tilde{\alpha}_j$. By induction on the number i_h of proximate points to q_1, it is easy to check that $\tilde{\alpha}_1 = \tilde{\alpha}_2 + \sum_{j=3}^{i_h} \tilde{\alpha}_j \equiv \tilde{\alpha}_2 \pmod 4$, thus $\tilde{\alpha}_1 + \tilde{\alpha}_2 \equiv 2 \pmod 4$ (cf. the genus formula (11.7)).

13 Some examples

Example 13.1 *Let B_0 be defined by: $y(y - x^2)(y + x^2) = 0$. One usually says that B_0 has two infinitely near triple points at q_1. Our combinatorial data are:*

$$\text{③} \longleftarrow \text{④} \qquad \textit{or equivalently} \qquad M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{matrix} \mu = (3,4), \\ \varepsilon = (1,0). \end{matrix}$$

The exceptional curves for $\tau : X \to X_0$ are a smooth rational curve F_1 with $F_1^2 = -1$ and a smooth elliptic curve F_2 with $F_2^2 = -2$ that meet in a point P. By Theorems 9.7 and 11.2, the fiber cycle of the canonical resolution is $F = 2F_1 + F_2$, while the fundamental cycle is $Z = F_1 + F_2 < F$, as one may also check directly. Moreover $F_{2|F_2} = -2P$ by (9.11).

The minimal resolution $\bar{\tau} : \bar{X} \to X_0$ is obtained by contracting the (-1)-curve F_1. Therefore $\bar{F}_2$ is the only exceptional curve for $\bar{\tau}$ and $\bar{F}_2$ is a smooth elliptic curve with $\bar{F}_2^2 = -1$. Clearly the fiber cycle and the fundamental cycle of the minimal resolution $\bar{\tau}$ are $\bar{Z} = \bar{F} = \bar{F}_2$.

The previous example can be generalized as follows.

Example 13.2 *Let B_0 be a curve with $2k$ infinitely near points q_1, ..., q_{2k} of the same odd multiplicity $\tilde{\alpha}_1 = \cdots = \tilde{\alpha}_{2k} = 2g + 1$, for some $g \geq 1$. More precisely, $q_i >^1 q_{i-1}$ for $1 < i \leq 2k$ and the weighted Enriques digraph is:*
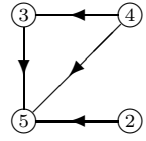
$$\boxed{2g+1} \longleftarrow \boxed{2g+2} \longleftarrow \boxed{2g+1} \longleftarrow \quad \cdots \quad \longleftarrow \boxed{2g+1} \longleftarrow \boxed{2g+2}$$

The exceptional curves for $\tau : X \to X_0$ are the following: (-1)-curves F_{2i-1}, for every $i = 1, \dots, k$; smooth rational curves F_{2i} with self-intersection -4, for $i = 1, \dots, k-1$, and a smooth curve F_{2k} of genus g with $F_{2k}^2 = -2$. By Theorems 9.7 and 11.2, the fiber cycle and the fundamental cycle of the canonical resolution are respectively:

$$F = \sum_{i=1}^{k} (2F_{2i-1} + F_{2i}), \qquad Z = F_1 + F_2 + \sum_{i=2}^{k} (2F_{2i-1} + F_{2i}).$$

Blow down the F_{2i-1}'s, for $i = 1, \ldots, k$, thus the exceptional curves for the minimal resolution $\bar{\tau} : \bar{X} \to X_0$ are the $\bar{F}_{2i}$'s, for $i = 1, \ldots, k$, which are smooth rational curves with self-intersection -2, except $\bar{F}_{2k}$ which is smooth of genus g with $\bar{F}_{2k}^2 = -1$. The fundamental cycle equals the fiber cycle of the minimal resolution $\bar{Z} = \bar{F} = \sum_{i=1}^{k} \bar{F}_{2i}$.

Example 13.3 *(cf. [19, p. 322]) Let B_0 be defined by: $y(x^4 + y^6) = 0$. In this case our combinatorial data are:*

$$M = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{matrix} \mu = (5, 2, 3, 4), \\ \varepsilon = (1, 0, 1, 0). \end{matrix}$$

The fiber cycle of the canonical resolution is $F = 2F_1 + F_2 + 2F_3 + 2F_4$, while the fundamental cycle is $Z = F_1 + F_2 + F_3 + F_4 < F$, as it should be by Theorem 12.2, because q_1 is a very odd vertex, its proximity digraph has just two petals (one of length 1 and the other of length 2) and q_3 is also very odd. The minimal resolution is obtained by blowing down F_3, therefore:

$$\bar{F} = 2\bar{F}_1 + \bar{F}_2 + 2\bar{F}_4 > \bar{F}_1 + \bar{F}_2 + \bar{F}_4 = \bar{Z}.$$

Rational double points (see [2], [11] and [4]). It is very well-known that the rational double points are given by the following equations:

$$\begin{aligned} A_n &: \quad z^2 = x^2 + y^{n+1}, & (n \geq 1), \\ D_n &: \quad z^2 = y(x^2 + y^{n-2}), & (n \geq 3), \\ E_6 &: \quad z^2 = x^3 + y^4, \\ E_7 &: \quad z^2 = x(x^2 + y^3), \\ E_8 &: \quad z^2 = x^3 + y^5 \end{aligned}$$

and the minimal resolution consists in smooth rational curves of self-intersection -2 whose dual graph is the corresponding Dynkin diagram.

Note that, for a rational double point, the fundamental cycle Z of the canonical resolution equals the fiber cycle F. Indeed if $Z < F$, then formula (11.7) says that $p_a(Z) = (\tilde{\alpha}_1 + \tilde{\alpha}_2 - 2)/4$, where $\tilde{\alpha}_1$ is odd and $\tilde{\alpha}_1 \geq 2$, so $p_a(Z) > 0$, contradicting Artin's criterion (which says that $p_a(Z) = 0$ if and only if the singularity is rational, cf. [2, Theorem 3]).

Moreover, starting from the well-known formula for the arithmetic genus of a sum of two curves, that is $p_a(C + D) = p_a(C) + p_a(D) + (C \cdot D) - 1$, and using (9.3) and (9.4) we find out that (cf. (9.12)):

$$p_a(Z) = p_a(F) = \frac{1}{2} \sum_{i=1}^{n} m_{1i}(\gamma_i - (\varepsilon_i - 2)E_i^2 - 4).$$

Hence we can see directly that $p_a(Z) = 0$ if and only if every branched exceptional divisor has self-intersection -4, every unbranched exceptional divisor has self-intersection -1 and $\gamma_i = 2$, or self-intersection -2 and $\gamma_i = 0$. Thus every F_i (if

F_i splits, every irreducible component of F_i) is rational with self-intersection -2 and the canonical resolution is minimal.

14 Adjunction conditions

We want to study the conditions that a double point singularity $p \in X_0$ imposes to canonical and pluricanonical systems of a surface. Recall that locally p is $\pi_0^{-1}(q_1)$, where $\pi_0 : X_0 \to Y_0$ is a double cover, Y_0 is a smooth surface, X_0 is normal and q_1 is an isolated singular point of the branch curve B_0 of π_0. Then we consider the canonical resolution $\pi : X \to Y$, that is a double cover branched along the smooth curve B.

In (9.6) we defined the adjunction condition divisor D^* as the pullback of

$$D = \sum_i (\alpha_i/2 - 1)E_i^* = \sigma^*(K_{Y_0} + B_0/2) - (K_Y + B/2). \tag{14.1}$$

So it suffices to understand what are the conditions that D imposes to the adjoint linear system $|K_Y + B/2|$. It is well known that $D = 0$, or equivalently $D^* = 0$, if and only if $p \in X_0$ is a rational double point (cf. previous section).

By applying σ_* to the exact sequence $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$, one sees that $\mathcal{I}_\Gamma := \sigma_* \mathcal{O}_Y(-D)$ is the ideal sheaf of a zero-dimensional scheme Γ supported at $q_1 \in Y_0$. Let us call $\mathcal{I}_\Gamma$ the *adjoint ideal* of the singularity.

For our convenience, let us assume that X_0 is a double plane, i.e., $Y_0 = \mathbb{P}^2$. Indeed the double point singularity is locally given as a double cover of an open disc, thus we may always find an irreducible plane curve B_0 of arbitrarily high (and even) degree whose germ at q_1 is analitically isomorphic to the germ of the branch curve of the double cover at q_1.

By (3.2), (3.3) and the projection formula we have that $\pi_* K_X \cong K_Y \oplus (K_Y + B/2)$, so $p_g(X) = h^0(X, K_X) = h^0(Y, K_Y + B/2)$ and $q(X) = h^1(X, K_X) = h^1(Y, K_Y + B/2)$. Riemann–Roch Theorem for $K_Y + B/2$ on Y and for $K_{\mathbb{P}^2} + B_0/2$ on $\mathbb{P}^2$ implies that:

$$h^0(K_X) - h^1(K_X) = B \cdot (K_Y + B/2)/4 + 1,$$
$$h^0(K_{X_0}) = h^0(K_{\mathbb{P}^2} + B_0/2) = B_0 \cdot (K_{\mathbb{P}^2} + B_0/2)/4 + 1.$$

It follows from (9.5) that

$$h^0(K_{X_0}) - h^0(K_X) + h^1(K_X) = \sum_{i=1}^n \frac{\alpha_i(\alpha_i - 2)}{8} =: c, \tag{14.2}$$

where c is defined by (14.2). Let us recall a well-known theorem of De Franchis:

Theorem 14.3 (De Franchis) *Let $\pi_0 : X_0 \to \mathbb{P}^2$ be a double plane and $\pi : X \to Y$ its canonical resolution. Then $q(X) > 0$ if and only if there is a plane curve B' (possibly $B' = 0$) such that:*

$$B_0 + 2B' = C_1 + C_2 + \cdots + C_m \tag{14.4}$$

where $C_1, \ldots, C_m$ are curves belonging to one and the same pencil and $m = 2q(X) + 2$ (resp. possibly $m = 2q(X) + 1$ if the pencil contains a double curve).

Proof. See [9] (or [8] for a modern proof). □

Corollary 14.5 *With the above notation, the number $h^0(K_{X_0}) - h^0(K_X)$ of conditions that the singularity $p \in X_0$ imposes to the canonical system is:*

$$c = \sum_{i=1}^{n} \frac{\alpha_i(\alpha_i - 2)}{8} = \sum_{i=1}^{n} \frac{\alpha_i/2(\alpha_i/2 - 1)}{2}. \tag{14.6}$$

Proof. Since we assumed B_0 to be irreducible, then $q(X) = h^1(K_X) = 0$ by De Franchis' Theorem and (14.6) follows from (14.2). □

We remark that De Franchis' Theorem allows us to compute the adjunction conditions even if B_0 were a given reducible curve and its degree were not assumed to be arbitrarily high.

We want to determine which singularity the general element C in $|\mathcal{I}_\Gamma(h)| = |\sigma_* \mathcal{O}_Y(K_Y + B/2)| = |\sigma_* (\sigma^*(\mathcal{O}_{\mathbb{P}^2}(hL)) \otimes \mathcal{O}_Y(-D))|$ has at q_1, where L is a general line in $\mathbb{P}^2$ and $h = \deg(B_0)/2 - 3$. According to formulas (14.1) and (14.6), one might expect that C has exactly multiplicity $\alpha_i/2 - 1$ at q_i, for every $i = 1, \dots, n$. The next example shows that this is not always the case.

Suppose that $q_1 \in B_0$ is the same singularity of Example 13.1. Since $\alpha = \mu - \varepsilon$, we have $\alpha_1 = 2$ and $\alpha_2 = 4$, thus one expects the general element C in $|\mathcal{I}_\Gamma(h)|$ to pass simply through q_2 and not to pass through q_1. But this is not possible, because q_2 is infinitely near to q_1.

Actually, we see that $(K_Y + B/2) \cdot E_1 < 0$ and E_1 is a fixed component of $|K_Y + B/2| = |\sigma^*(hL) - E_2|$. Moreover

$$K_Y + B/2 - E_1 = \sigma^*(hL) - 1 \cdot E_1^* - 0 \cdot E_2^* \tag{14.7}$$

meets non negatively E_1 and E_2, so $|K_Y + B/2 - E_1|$ has no fixed components by the next Lemma 14.8. Formula (14.7) means that C passes simply through q_1 and does not pass through q_2 (which is 1 adjunction condition as well).

Lemma 14.8 *Let $\mathcal{L}$ be a linear system on Y which we write as:*

$$\mathcal{L} = \left| \sigma^*(hL) - \sum_{i=1}^{n} m_i E_i^* \right|,$$

where L is a general line in $\mathbb{P}^2$, m_i are non-negative integers and h is arbitrarily high. Suppose that $\deg \mathcal{L}_{|E_i} \geq 0$, for every $i = 1, \dots, n$. Then $\mathcal{L}$ has no fixed component. In particular the general member of $\sigma_* \mathcal{L}$ is a plane curve with multiplicity exactly m_i at q_i, for $i = 1, \dots, n$.

Proof. Since $h \gg 0$, we may assume that the only possible fixed components of $\mathcal{L}$ are among the E_i's. For every $i = 1, \dots, n$, consider the exact sequence $0 \to \mathcal{L}(-E_i) \to \mathcal{L} \to \mathcal{L}_{|E_i} \to 0$. We need to show that $h^0(\mathcal{L}(-E_i)) < h^0(\mathcal{L})$. This will follow from $H^1(\mathcal{L}(-E_i)) = 0$, because $H^0(\mathcal{L}_{|E_i}) \neq 0$ by assumption. We claim that $R^1 \sigma_* \mathcal{L}(-E_i) = 0$. This will imply that $H^1(\mathcal{L}(-E_i)) = H^1(\sigma_* \mathcal{L}(-E_i)) = 0$,

where the last equality follows from Serre's Theorem, because $h \gg 0$, and we will be done. Indeed $H^1(\mathcal{L}(-E_i)_{|E_1^*}) = 0$, since E_1^* is 1-connected, $p_a(E_1^*) = 0$ and $\deg \mathcal{L}(-E_i)_{|E_1^*} \geq m_1 \geq 0$. $\qquad\square$

The above discussion suggested us to introduce the following notion: we say that a point q_i is *defective* if there exists a point q_j such that $\alpha_j > \alpha_i$ and q_j is infinitely near of order one to q_i. Hence, if q_i is defective, then $B \cdot E_i < 0$, while we know that $\tilde{B} \cdot E_i \geq 0$ for every $i = 0, \dots, n$, because $\tilde{B}$ is the proper transform of a plane curve. In Example 13.1 (recalled before the previous lemma), q_1 is defective, because $q_2 >^1 q_1$ and $4 = \alpha_2 > \alpha_1 = 2$.

Lemma 14.9 *A point q_i is defective if and only if $D \cdot E_i > 0$. More precisely, a point q_i is defective if and only if $\varepsilon_i = 1$ and there exists a (necessarily unique) point $q_j >^1 q_i$ with $\tilde{\alpha}_j = \tilde{\alpha}_i$ and $\varepsilon_j = 0$. Furthermore, either:*

(i) $\alpha_i = \tilde{\alpha}_i - 1$, or
(ii) $\alpha_i = \tilde{\alpha}_i$ and both q_i, q_j are proximate to a point q_k with $\varepsilon_k = 1$.

Finally a point q_i is defective if and only if F_i is a (-1)-curve.

Proof. The last statement follows easily from the other ones and Remark 10.3. By definition, if q_i is defective, then $D \cdot E_i \geq -(B \cdot E_i)/2 > 0$. Conversely, $D \cdot E_i \leq 0$ is equivalent to $(B + 2K_Y) \cdot E_i \geq 0$, that holds, if q_i is not defective, by Lemma 4.3 in [6]. This proves the first statement.

Note that if there is a point $q_j >^1 q_i$ with $\tilde{\alpha}_j = \tilde{\alpha}_i$, then q_j is the unique proximate point to q_i.

Suppose that $q_i >^1 q_k$ and q_k is the only point which q_i is proximate to. Then $\alpha_i = \tilde{\alpha}_i + \varepsilon_k - \varepsilon_i$ by (7.3). Let q_j be an infinitely near point of order one to q_i. If q_j is proximate only to q_i, then $\alpha_j = \tilde{\alpha}_j + \varepsilon_i - \varepsilon_j$, thus $\alpha_j \geq \alpha_i + 2$ if and only if

$$(\tilde{\alpha}_i - \tilde{\alpha}_j) + \varepsilon_k + \varepsilon_j + 2 \leq 2\varepsilon_i,$$

which (recalling that $\tilde{\alpha}_i \geq \tilde{\alpha}_j$ because $\tilde{B} \cdot E_i \geq 0$) holds only if $\tilde{\alpha}_i - \tilde{\alpha}_j = \varepsilon_k = \varepsilon_j = 0$ and $\varepsilon_i = 1$, that is case (i).

If q_j is proximate also to q_k, then $\alpha_i = \tilde{\alpha}_j + \varepsilon_i + \varepsilon_k - \varepsilon_j$, hence $\alpha_j \geq \alpha_i + 2$ if and only if $(\tilde{\alpha}_i - \tilde{\alpha}_j) + \varepsilon_j + 2 \leq 2\varepsilon_i$, that is either case (i) or (ii) depending on the value of ε_k.

This concludes the proof in case q_i is proximate to only one point. One may proceed similarly for the other configurations of $q_j >^1 q_i$, namely if q_i is not infinitely near to any point or if q_i is proximate to more than one point. $\qquad\square$

Both of the cases (i) and (ii) of Lemma 14.9 may occur, as the point q_1 in Example 13.1 and the point q_3 in Example 13.3 respectively show.

We remark that if q_i is defective and q_j is as above, namely $q_j >^1 q_i$ and $\alpha_j > \alpha_i$, then q_j cannot be defective. However there may exist a defective point q_l with $q_l >^1 q_j$ and $\alpha_l = \alpha_i$, as $q_3, q_5, \dots, q_{2k-1}$ in Example 13.2.

We say that a point q_i is *1-defective*, and we write $\mathrm{def}(q_i) = 1$, if q_i is defective and there is no defective point $q_j > q_i$ with $\alpha_j = \alpha_i$. Inductively, we say that q_i

is k-*defective*, and we write $\operatorname{def}(q_i) = k$, if there exists a $(k-1)$-defective point $q_j >^2 q_i$ with $\alpha_j = \alpha_i$.

In Example 13.2, the point q_1 is k-defective. We set $\operatorname{def}(q_i) = 0$ if q_i is not defective and $\operatorname{Def} = \{i \mid \operatorname{def}(q_i) > 0\}$. Thus $i \in \operatorname{Def}$ if and only if q_i is defective.

Now we are ready to show what exactly happens to an element in $|\mathcal{I}_\Gamma(h)|$ at a defective point. To simplify the notation, by ordering conveniently the blowing-ups, we may and will assume that if q_j, q_k are defective, with $\alpha_j = \alpha_k$ and $q_k >^2 q_j$, then $k = j + 2$ and $q_{j+2} >^1 q_{j+1} >^1 q_j$.

Theorem 14.10 *The fixed part of $|K_Y + B/2|$ is exactly:*

$$\bar{E} = \sum_{j \in \operatorname{Def}} \sum_{r=0}^{\operatorname{def}(q_j)-1} E_{j+r}. \tag{14.11}$$

Proof. By Lemma 14.9, the non-defective points do not mind, so we may focus only on what happens at a defective point. Any k-defective point looks like the point q_1 of Example 13.2, thus we will assume that q_1 is k-defective and we will follow the notation of that example. Recall that, for every $i = 1, \ldots, k$, the point q_{2i-1} is $(k-i+1)$-defective, $\alpha_{2i}/2 - 1 = g$ and $\alpha_{2i-1}/2 - 1 = g - 1$.

We claim that the fixed part of $|K_Y + B/2|$ is exactly:

$$\bar{E} = \sum_{l=0}^{k-1} \sum_{i=1}^{k-l} E_{2i+l-1} = \sum_{j \in \operatorname{Def}} \sum_{r=0}^{\operatorname{def}(q_j)-1} E_{j+r}, \tag{14.12}$$

and formula (14.11) clearly follows. Note that (14.12) means that the general element of $|\mathcal{I}_\Gamma(h)|$ has multiplicity g at $q_1, \ldots, q_k$ and $g-1$ at $q_{k+1}, \ldots, q_{2k}$, giving kg^2 adjunction conditions as expected.

Now we prove our claim. By Lemma 14.9, the exceptional curve E_{2i-1} is a fixed component of $|K_Y + B/2|$, for $i = 1, \ldots, k$. Then

$$D + \sum_{i=1}^{k} E_{2i-1} = \sum_{j=1}^{k} \left(g E^*_{2j-1} + (g-1) E^*_{2j} \right)$$

meets positively E_{2j} for $j = 1, \ldots, k-1$, which therefore are fixed components of $|K_Y + B/2|$ too. Now

$$D + \sum_{i=1}^{k} E_{2i-1} + \sum_{j=1}^{k-1} E_{2j} = g E^*_1 + \sum_{l=1}^{k-1} \left(g E^*_{2l} + (g-1) E^*_{2l+1} \right) + (g-1) E^*_{2k}$$

meets again positively E_{2i-1}, for $i = 2, \ldots, k$ (but not E_1 and E_{2k}).

Going on in this way, by induction on k, it follows that $|K_Y + B/2|$ contains (14.12). On the other hand, $D + \bar{E} = \sum_{i=1}^{k} (g E^*_i + (g-1) E^*_{k+i})$ does not meet positively anyone of the E_i's, thus the fixed part of $|K_Y + B/2|$ is exactly $\bar{E}$, by Lemma 14.8, and our claim is proved. $\square$

We remark that the previous theorem gives an alternative proof of Corollary 14.5, independent from De Franchis' Theorem.

Finally we want to compute the number of conditions that the singularity $p \in X_0$ imposes to *pluricanonical* systems. The plurigenera of X are:

$$P_m(X) = h^0(X, mK_X) = h^0(Y, mK_Y + mB/2) + h^0(Y, mK_Y + (m-1)B/2).$$

Riemann–Roch Theorem and (9.5) imply that:

$$h^0(mK_{X_0}) - h^0(mK_X) + h^1(mK_X) = \sum_{i=1}^{n} \frac{2(m^2 - m)(\alpha_i - 2)^2 + \alpha_i^2 - 2\alpha_i}{8}.$$

Theorem 14.13 *The number of conditions $h^0(mK_{X_0}) - h^0(mK_X)$ that the singularity $p \in X_0$ imposes to the m-canonical system are:*

$$\sum_{i=1}^{n} \frac{2(m^2 - m)(\alpha_i - 2)^2 + \alpha_i^2 - 2\alpha_i}{8} - \frac{dm(m-1)}{2} \qquad (14.14)$$

where $d := \sharp\mathrm{Def}$ is the number of defective points.

Proof. We shall show that $h^1(mK_X) = dm(m-1)/2$. Since we are dealing with local questions, we may assume that the (-1)-curves of X are contained in $\tau^{-1}(p)$. Recall that these (-1)-curves are disjoint and there are exactly d of them. Let $\tau' : X \to \bar{X}$ be their contraction (see section 10), then $\bar{X}$ is a minimal surface of general type and we may assume that $h^0(K_{\bar{X}}) \gg 0$ (because h is arbitrarily high). With no loss of generality, we may also assume that τ' is the blowing-down of just a (-1)-curve F_i. Thus it suffices to show that, under these assumptions, $h^1(mK_X) = m(m-1)/2$. By Serre duality, $h^1(mK_X) = h^1(-(m-1)K_X)$. Let C be a curve in $|(m-1)K_X|$. Clearly $C = \tau'^*(C_0) + (m-1)F_i$, where $C_0 \in |(m-1)K_{\bar{X}}|$. It is well-known that C_0 and $\tau'^*(C_0)$ are 1-connected (see [4, Proposition 6.1] and [14, §1]), therefore $h^0(\mathcal{O}_{\tau'^*(C_0)}) = 1$. Since $q(X) = h^1(\mathcal{O}_X) = 0$, the exact sequence of sheaves $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ implies that $h^1(\mathcal{O}_X(-C)) = h^0(\mathcal{O}_{(m-1)F_i})$. Finally one easily checks that $h^0(\mathcal{O}_{(m-1)F_i}) = m(m-1)/2$. $\square$

We remark that, if h is not assumed to be arbitrarily high, $\bar{X}$ may not be of general type and one should compute, or estimate, $h^1(mK_X)$.

As we did for $|K_Y + B/2|$, we want to determine the fixed components of $|mK_Y + mB/2|$ and $|mK_Y + (m-1)B/2|$. After having ordered the blowing-ups as explained just before Theorem 14.10, we are ready to prove the following:

Theorem 14.15 *The fixed part of $|mK_Y + \bar{m}B/2|$, for $\bar{m} = m$ or $\bar{m} = m - 1$, is exactly:*

$$\left[\frac{\bar{m}}{2} \right] \sum_{j \in \mathrm{Def}} E_j + (\bar{m} \bmod 2)\bar{E}, \qquad (14.16)$$

where $\bar{E}$ is (14.11), $[\bar{m}/2]$ is the largest integer smaller than or equal to $\bar{m}/2$ and $\bar{m} \bmod 2 = \bar{m} - 2[\bar{m}/2] \in \{0, 1\}$. $\square$

Proof. As in the proof of Theorem 14.10, it suffices to understand what happens at a defective point, thus we assume that q_1 is a k-defective point as in Example 13.2. Let us set

$$\tilde{E} = \sum_{i=1}^{k} E_{2i-1} = \sum_{j \in \mathrm{Def}} E_j.$$

Suppose that $\bar{m} = m$. If m is even, then

$$mD = \sum_{i=1}^{k} \left(m(g-1)E_{2i-1}^* + mgE_{2i}^* \right),$$

hence mD meets positively E_{2i-1} for every $i = 1, \ldots, k$. Moreover $(mD + j\tilde{E}) \cdot E_{2i-1} > 0$ for every $i = 1, \ldots, k$ and $j = 1, \ldots, m/2 - 1$. This means that $m\tilde{E}/2$ is a fixed component of $|mK_Y + mB/2|$. Then

$$mD + \frac{m}{2}\tilde{E} = \sum_{i=1}^{2k} \left(mg - \frac{m}{2} \right) E_i$$

does not meet positively anyone of the E_i's, therefore the fixed part of $|mK_Y + mB/2|$ is exactly $m\tilde{E}/2$, by Lemma 14.8.

If m is odd, following the same argument, one easily shows that the fixed part of $|mK_Y + mB/2|$ is $(m-1)\tilde{E}/2 + \bar{E}$.

One proceeds similarly in the case that $\bar{m} = m - 1$. Indeed, one can prove that the fixed part of $|mK_Y + (m-1)B/2|$ is $(m-1)\tilde{E}/2$ (resp. $m\tilde{E}/2 + \bar{E}$) if m is odd (resp. if m is even). □

Note that Theorem 14.13 can be proved also as corollary of Theorem 14.15.

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